BUCKLING/POSTBUCKLING ANALYSIS OF RECTANGULAR ORTHOTROPIC PLATES

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CERTIFICATE

This is to certify that the research carried out by Suresh Chand Mittal for the preparation of the thesis, 'Buckling/Postbuckling Analysis of Rectangular Orthotropic Plates', has been supervised by me. This thesis is being submitted to the Department of Civil Engineering, Indian Institute of Technology, Kanpur, in partial fulfillment of the requirements for the Degree of Master of Technology, and has not been submitted for a degree elsewhere.

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TABLE OF CONTENTS

PAGE

LIST OF FIGURES

NOMENCLATURE

All symbols are defined whenever they appear first in the text.

a	–	plate length
b	–	plate width
b_e	–	effective width of plate
c_x	–	centroidal axis distance of stiffened element from mid plane of plate
C_1, C_2, C_3	–	Constants equal to $E_x h/(1-\nu_x \nu_y)$, $E_y h/(1-\nu_x \nu_y)$, $G_{xy} h$, respectively
d_x	–	spacing of stiffeners
D_x, D_y	–	flexural rigidities of plate in x-and y-directions, equal to $E_x h^3/(1-\nu_x \nu_y)$, and $E_y h^3/(1-\nu_x \nu_y)$, respectively
D_{xy}	–	torsional rigidity of plate = $G_{xy} h^3/6$
E	–	Young's modulus for plate material
E_x, E_y	–	Young's modulus for plate material in x- and y-directions , respectively
E^*	–	E_x/E_y
$F(x,y)$	–	stress function, defined on page 58
G_{xy}	–	shear modulus of rigidity
h	–	plate thickness
$\bar{h}$	–	equivalent plate thickness
h_s	–	equivalent plate thickness (given by IS:801-1975)
I_n	–	moment of inertia of stiffener about the neutral axis of plate

I_s	-	moment of inertia of the full area of multiple-stiffened element, including intermediate stiffeners, about its own centroidal axis
J_x	-	torsional constant
K	-	critical buckling stress coefficient
m,n	-	numbers of half waves in which, plate buckles in x- and y-directions, respectively
M	-	$m\pi/a$
M_x, M_y	-	bending moments per unit width
M_{xy}	-	twisting moment per unit width
N	-	$n\pi/b$
N_x, N_y	-	resultant normal forces in x-and y-directions, respectively
N_{xy}	-	resultant shearing force in xy-plane
N_{cr}	-	buckling load per unit width
P	-	total compressive load in x-direction
P_o	-	buckling or critical load
s	-	perimeter of plate cross-section over width b
u,v,w	-	displacements in x-,y-, and z-directions, respectively
w_s	-	whole width of plate between webs (or from a web to edge stiffener)
α	-	arbitrary parameter = $[(P-P_o)/P_o]^{1/2}$
ϵ_x, ϵ_y	-	strains in x- and y-directions, respectively
γ_{xy}	-	shear strain in xy-plane

Δ - plate shortening in x-direction

Δ_o - plate shortening in x-direction at buckling load

ν - Poisson's ratio

ν_x, ν_y - Poisson's ratios for stiffened plate, in x-and y-directions, respectively

σ_x, σ_y - resulting stresses in x- and y- directions, respectively

σ_{xa} - resulting average stress in x-direction

σ_{max} - maximum resulting stress along loaded edges

σ_{cr} - resulting stress at buckling load, in x-direction.

λ - mb/a

ABSTRACT

The formula (IS:801-1975) for the equivalent thickness of a multiple stiffened compression element, used in the design of cold-formed structural members, is reviewed. By carrying out an analysis for the buckling strength of an orthotropic plate, a more rational formula is suggested for design.

Postbuckling analysis of a simply supported orthotropic plate is done using the method of successive approximation. In doing so, von Karman's large deflection equations are converted into an infinite set of linear differential equations by expanding the displacement into a power series in terms of an arbitrary parameter. The curves of critical buckling load, load-shortening and effective width are presented for plates having different elastic properties. The possibility of a change in the buckling pattern of the plate is also indicated.

Postbuckling analysis is also carried out by an approximate method similar to the one used by, for example, Chajes (1974) for isotropic plates.

CHAPTER I

INTRODUCTION

1.1 Buckling, Postbuckling and Effective Width Concepts:

Thin plate elements used in structures, are often subjected to normal and shearing forces acting in the plane of the plate. If these in-plane forces are sufficiently small, the equilibrium is stable and the resulting deformations are characterized by the absence of lateral displacements ($w=0$, $u\neq 0$, $v \neq 0$). As the magnitude of these in-plane forces increases, at a certain load intensity, a marked change in the character of the deformation pattern takes place. That is, simultaneously with the in-plane deformations, lateral displacements are introduced. In this condition, the originally stable equilibrium becomes unstable and the plate is said to have buckled. The load producing this condition is called the critical load. The importance of the critical load is the initiation of a deflection pattern, which, if the load is further increased, rapidly leads to large lateral deflections and eventually to complete failure of the plate. This is a dangerous condition which must be avoided.

In the mathematical formulation of elastic stability problems of plates, one uses the neutral equilibrium assuming

a bifurcation of the deformations . That is, at critical load, of the possible two paths of deformation (one associated with the stable equilibrium, and other one pertinent to the unstable equilibrium condition), the plate always takes the buckled form. In addition to the existence of this bifurcation of equilibrium paths, elastic stability analysis of plates assumes the validity of Hooke's law.

Classical buckling problems of plates can be formulated using (i) equilibrium approach which leads to an eigenvalue problem, (ii) energy methods such as Ritz method, Rayleigh's method, Galerkin's method, (iii) numerical methods using finite-difference and finite element techniques, and (iv) dynamic approach. Methods (i)-(iii) have been utilized in solving buckling problems for isotropic, stiffened and orthotropic plates by many investigators; a comprehensive account is available in, for example, Chajes (1974) and Szilard (1974). The use of dynamic approach (Leipholz, 1970) becomes a must when the plate is subjected to nonconservative forces.

Besides this classical buckling theory, the behaviour of flat plates after buckling is of considerable practical interest. The postbuckling analysis of plates is usually difficult, since it is basically a nonlinear problem and is markedly different from that of

thin struts. While a small increase in the critical load for struts will produce a complete collapse, the critical load may be but a small fraction of the load that causes failure in plates. This increased load-carrying capacity, beyond the critical load, originates from the effect of large deflections and from the fact that the longitudinal edges of the plate are usually constrained to remain straight. The use of this additional strength is of great practical importance in the design of structures, since by considering the postbuckling behaviour of plates, considerable weight savings can be achieved.

To study the postbuckling behaviour of thin plates, one has to go for the solutions of von Karman's large-deflection equations. Since the exact solutions of von Karman's equations are cumbersome, generally approximate methods have been employed, for example, by Timoshenko and Gere (1961) and Marguerre and Volmir (See, e.g.,Chajes, 1974). Wang (1949) and Brown and Harvey (1969) have used finite difference technique to solve large-deflection equations for plate subjected to uniform lateral pressure and compressive edge loading. Similar investigation has been made by Hooke (1969) using the perturbation method. The perturbation technique has also been applied by Stein (1959) in studying postbuckling behaviour of isotropic plate subjected to longitudinal compression. Rushton(1970)

has studied the same problem using the method of dynamic relaxation. For orthotropic plates, the only solution available is due to Prabhakara and Chia (1973) who have solved von Karman's equations by expressing displacements and the stress function in double Fourier series.

Till the buckling load is reached, the stress distribution along the loaded edges is uniform and along the unloaded edges its intensity is zero. As the load is increased beyond the critical value, the distribution of stress along the edges becomes progressively nonlinear. Tests have shown that the loaded edges are more heavily stressed in the vicinity of the unloaded edges and that stresses remain virtually unchanged near the centre of the loaded edges (Fig. 1-1(a)). The most common method of accounting for nonuniform changing stress distribution is to use the method of effective width, originally proposed by von Karman (1932). When failure of the plate is impending, almost the total compressive force is carried by two strips, each of width $b_e/2$, adjecent to the unloaded edges (Fig.1-1(b)). The quantity b_e, the effective width is defined as

$$b_e = \frac{1}{\sigma_{max}} \int_o^b \sigma_x . dy \qquad (1.1)$$

This implies that the centre portion of the plate is ineffective for the purpose of calculating the strength

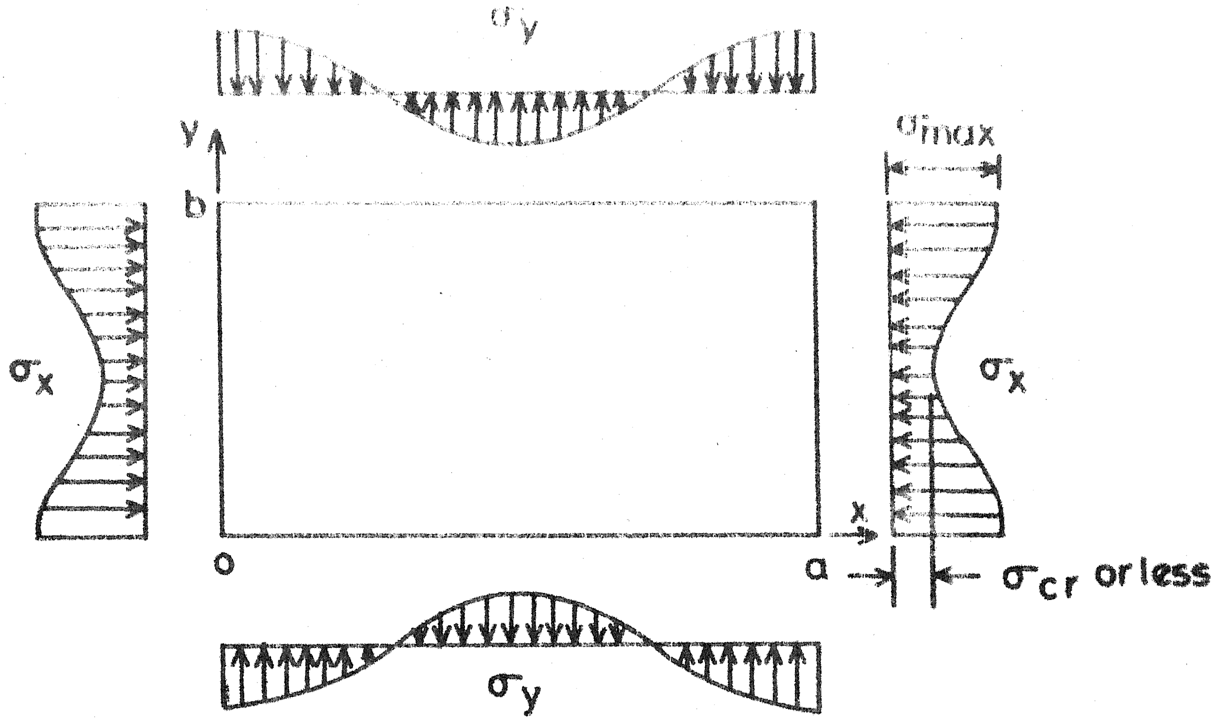

Fig. 1-1 (a) Stress distribution in postbuckling range.

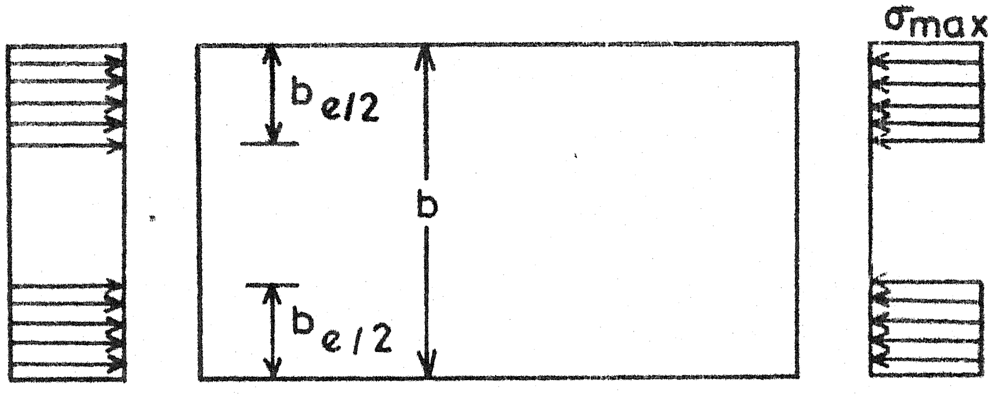

Fig. 1-1 (b) Effective width concept for determining postbuckling strength.

of the plate. With this concept, the problem becomes one of finding the effective width b_e at a specified edge stress σ_{max}. When b_e is known, the following relationships apply

(i) $\sigma_{max} \leq \sigma_{cr}$: $\sigma_{max} = \sigma_{av}$, $b_e = b$, $P = \sigma_{av}\, bh$

(ii) $\sigma_{max} > \sigma_{cr}$: $\sigma_{max}\, b_e = \sigma_{av} b$, $P = \sigma_{av} bh = \sigma_{max}\, b_e h$

(iii) $\sigma_{max} > \sigma_y$: $\sigma_{av} = \sigma_{ult}$, $b_e = b_{ult}$, $\sigma_y\, b_{ult} = \sigma_{ult} b$

$$P_{ult} = \sigma_{ult} bh = \sigma_y\, b_{ult}\, h$$

In these relations σ_{av} is the average stress due to the applied total load P, σ_{ult} is the value of σ_{av} at collapse, σ_y is the yield stress of plate material, P_{ult} is the value of P at collapse and b_{ult} is the value of b_e at collapse.

The effective width clauses in current Americal Specification (AISI, 1968)* and the earlier Canadian Standard (CSA-S136,1963) were based primarily on experimental investigations by Winter (1947), whose effective width formula is incorporated in these specifications. The current Canadian Standard (CSA-S136, 1974) contains an effective width formula of the type proposed by von Karman in 1932. This change in the Canadian code was

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* IS: 801-1975 specifications are also similar.

based on the work of Lind et al.(1976) who did a statistical analysis of the available experimental data. An excellent review of the experimental work concerning the effective width concept is given by Roorda and Venkataramaiah(1978).

1.2 Present Work:

Except the work of Prabhakara and Chia (1973), no theoretical investigation is available which deals with the postbuckling analysis of orthotropic plates. Moreover, the said work neither gives any relation for the effective width nor takes into account the effect of changes in the buckling pattern. In this thesis the postbuckling analysis of orthotropic rectangular plates of symmetric cross-section is carried out. Von Karman's large-deflection equations, as applied to orthotropic plates, are reduced to an infinite set of linear differential equations by expanding the displacements into a power series in terms of an arbitrary parameter. The first three equations of the infinite set correspond to small deflection theory whose solution yields the buckling load. Solution of the suceeding equations gives the postbuckling solutions. The curves of load-shortening and effective width are presented for plates having different elastic properties. The results obtained are compared with those of Stein (1959) for isotropic plates. Postbuckling analysis is also

carried out by an approximate method, similar to that given in Chajes (1974).

IS:801-1975 formula concerning the equivalent plate thickness, for design of multiple-stiffened light gauge sections, is also reviewed. A more rational formula is suggested on the basis of the buckling strength of orthotropic plates.

CHAPTER II

MULTIPLE STIFFENED LIGHT GAUGE COMPRESSION MEMBERS

2.1 Introduction:

Multiple stiffened compression elements are common in light gauge members made of steel or aluminium. The Indian Standard IS: 801-1975 specifications* for the design of cold-formed steel structured members specify that a flat compression element, to be considered multiple-stiffened, shall be stiffened between webs (or between a web and a stiffened edge) by intermediate stiffeners parallel to the direction of stress. If the stiffeners are spaced so closely that the sub-elements between stiffeners individually do not exceed the specified limit of flat width to thickness ratio for fully effective stiffened compression elements, 'the flat width ratio' for the entire element may be computed as that of an 'equivalent' fictitious, isotropic plate element without intermediate stiffeners whose width w_s is the whole width between webs (or from a web to the edge stiffener) and whose thickness h_s is determined from

$$h_s = \left(\frac{12\ I_s}{w_s} \right)^{1/3} \qquad (2.1)$$

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* The AISI and CSA standard specifications are also similar.

where I_s is the moment of inertia of the full area of the multiple-stiffened element, including the intermediate stiffeners, about its own centroidal axis. Infact the value of h_s specified in Eq. (2.1) is the thickness of a solid plate element having the same amount of moment of inertia about its centroidal axis as the actual element.

A closer examination of the mechanical behaviour of the element in compression reveals that the formula in Eq. (2.1) is based on a fallacy. Introduction of intermediate stiffeners transforms the plate into an orthotropic plate (Fig. 2.1) having much greater stiffness in the longitudinal direction than that in the transverse direction while the torsional stiffness is not very much different than that of the unstiffened plate. Since all the three modes are present in the buckled element, it is clear that Eq. (2.1) overestimates the thickness of the equivalent plate. In this chapter, a more rational formula for the equivalent thickness of a multiple-stiffened compression element is suggested which is based on the analysis of the buckling strength of an orthotropic plate.

2.2 Basic Equations:

To determine the critical in-plane loading of a flat plate, it is necessary to have the equations of

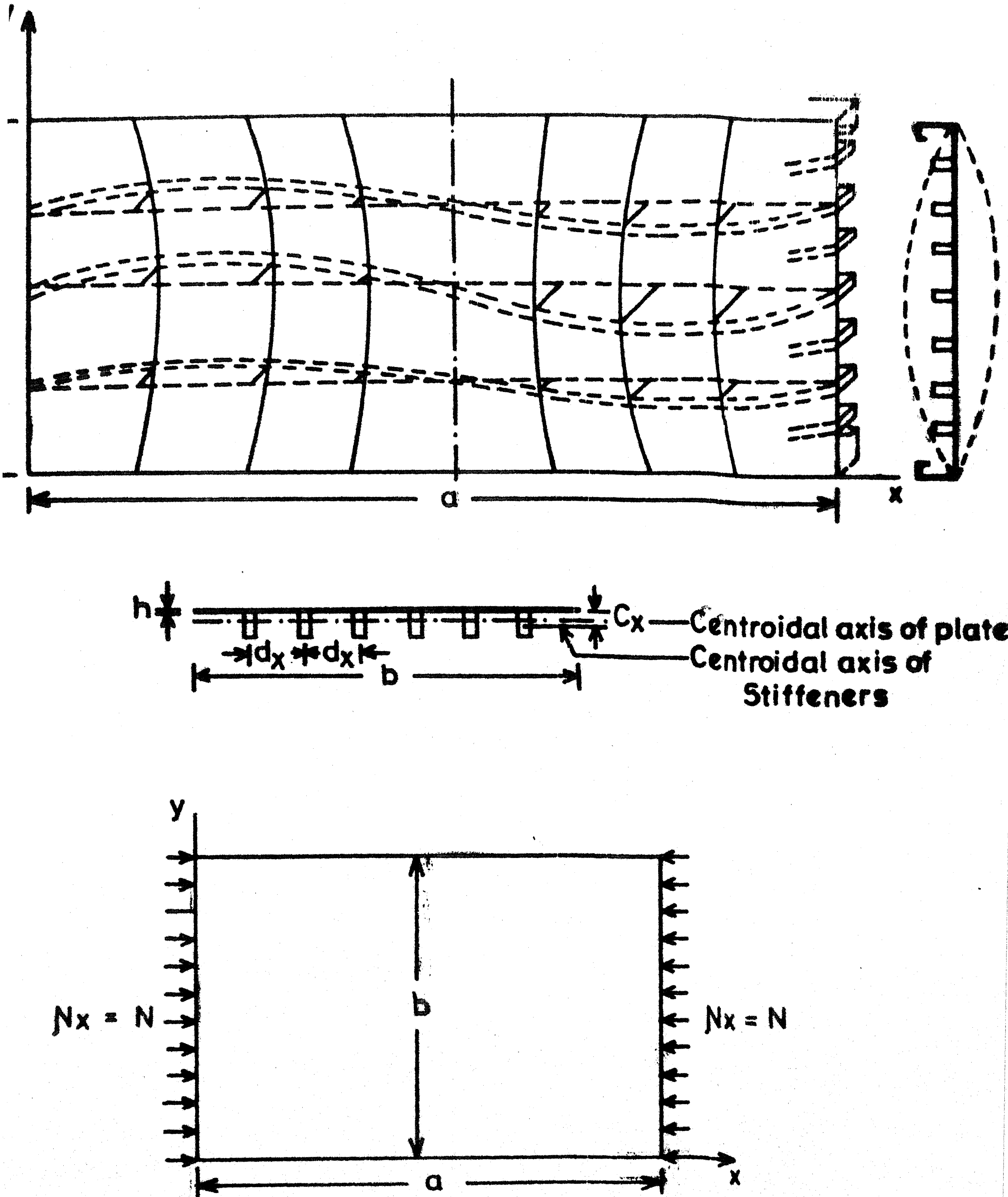

a
x
h
C_x
Centroidal axis of plate
d_x
d_x
b
Centroidal axis of Stiffeners
y
$N_x = N$
b
$N_x = N$
a
x

Fig. 2-1

equilibrium for the plate in a slightly bent configuration. For a loading condition consisting of constant biaxial compression forces (N_x and N_y) and constant in-plane shears (N_{xy}), this equation is expressed in the following form in terms of moments M_x, M_y, M_{xy} and the lateral displacement $w(x,y)$ of the middle surface of the plate (cf., e.g., Timoshenko (1961)):

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{2\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \tag{2.2}$$

Equations that express the moments in terms of displacements will be obtained by relating the moments to the stresses, the stresses to strains, and finally the strains to the displacement. For a thin orthopropic elastic plate, these relations can be written as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = - \begin{bmatrix} D_x & \nu_y D_x & 0 \\ \nu_x D_y & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ -\partial^2 w/\partial x \partial y \end{Bmatrix} \tag{2.3}$$

where

$$D_x = \frac{E_x h^3}{12(1-\nu_x\nu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\nu_x\nu_y)}, \quad D_{xy} = \frac{G_{xy}h^3}{6} \tag{2.4}$$

Here, the plate is of uniform thickness h. The quantities D_x, E_x, ν_x are, respectively, the flexural rigidity per unit width of plate, the Young's modulus and the Poisson's ratio in the x-direction while the corresponding quantities in the y-direction are D_y, E_y and ν_y. D_{xy} is the torsional stiffness and the shear modulus G_{xy} is expressed as

$$G_{xy} = \frac{E_x E_y}{E_x + E_y + 2\nu_x E_y}$$

Further due to Betti's reciprocal theorem

$$\nu_x D_y = \nu_y D_x = \nu D \qquad (2.5)$$

in which ν and D are the Poisson's ratio and the flexural rigidity for an isotropic plate. Now, the substitution of (2.3) and (2.5) into (2.2) gives, the desired differential equation:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(\nu D + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4}$$

$$-N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \qquad (2.6)$$

2.3 Buckling Strength:

We idealize the problem by assuming that the edge flanges of the member prevent out-of-plane displacement of the edges of the plate. It is also assumed that the

flanges offer negligible resistance against rotation and in-plane displacement. Then the boundary conditions may be written as

$$N_y = N_{xy} = 0,\ w = \frac{\partial^2 w}{\partial y^2} = 0;\ y=0,b;\ 0 < x \leq a \qquad (2.7)$$

The ends of the plate are assumed to be restrained against out-of-plane displacements but force to move in the plane, and loaded by a constant compressive force of intensity N per unit length of mid surface:

$$N_x = -N,\ N_{xy} = 0,\ w = \frac{\partial^2 w}{\partial x^2} = 0;\ x = 0,\ a;\ 0 < y \leq b \qquad (2.8)$$

Let us assume that the solution to Eq. (2.6) (with $N_y = N_{xy} = 0$) is of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (2.9)$$

in which m and n are the number of half-waves that the member buckles into in the x and y-directions, respectively. The assumed solution satisfies the boundary conditions (2.7) and (2.8), and there remains only the task of ensuring that it also satisfies the differential equation. Substitution of (2.9) into (2.6) eventually leads to an eigenvalue problem; for a nontrivial solution we must have

$$N = \left(\frac{a}{m\pi}\right)^2 \left[D_x \left(\frac{m\pi}{a}\right)^4 + 2(\nu D + D_{xy}) \quad \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + D_y \left(\frac{n\pi}{b}\right)^4 \right] \tag{2.10}$$

To determine the least value of N, the expression on the r.h.s. of (2.10) should be minimize w.r.t. m and n. It is found that N is minimum when n = 1 and m takes the value

$$m = \frac{a}{b}\left(\frac{D_y}{D_x}\right)^{1/4} \tag{2.11}$$

Consequently, from (2.10) we obtain

$$N_{cr} = \frac{2\pi^2}{b^2}\left[(D_x D_y)^{1/2} + \nu D + D_{xy} \right] \tag{2.12}$$

For an isotropic plate, since

$$D_x = D_y = \frac{D_{xy}}{(1-\nu)} = D = \frac{Eh^3}{12(1-\nu^2)}$$

(2.12) yields the familiar result:

$$N^i = \frac{4\pi^2 D}{b^2} \tag{2.12a}$$

2.4 Equivalent Plate Thickness:

From (2.12) and (2.12a) it is obvious that

$$N_{cr} = \frac{1}{2}\left[\frac{(D_x D_y)^{1/2}}{D} + \nu + \frac{D_{xy}}{D} \right] N^i \tag{2.13}$$

It follows that the orthotropic plate has the same buckling load as of an isotropic plate of the same material and width and having the thickness $\bar{h}$ given by

$$\bar{h} = h \left[\frac{1}{2} \left\{ \left(\frac{D_x D_y}{D^2}\right)^{1/2} + \nu + \frac{D_{xy}}{D} \right\} \right]^{1/3} \qquad (2.14)$$

For a stiffened plate (Fig. 2.1), various rigidities can be evaluated in the following manner (cf., e.g., Szilard (1974)):

$$\left.\begin{aligned} D_y &\simeq D = \frac{Eh^3}{12(1-\nu^2)} \\ D_x &\simeq D + \frac{Ehc_x^2}{1-\nu^2} + \frac{EI_n}{d_x} \\ (\nu D + D_{xy}) &\simeq D + \frac{E}{4(1+\nu)} \frac{J_x}{d_x} \end{aligned}\right\} \qquad (2.15)$$

In (2.15), J_x is the torsional constant, I_n is the moment of inertia of stiffener about the neutral axis of the plate and c_x , d_x are explained in the Fig. (2-1). Use of (2.15) in (2.14) results

$$\frac{\bar{h}}{h} = \left[\frac{1}{2} \left\{ \left(1 + \frac{12c_x^2}{h^2} + \frac{12(1-\nu^2)I_n}{d_x h^3}\right)^{1/2} + 1 + 3(1-\nu) \frac{J_x}{d_x h^3} \right\}\right]^{1/3}$$

(2.17)

Alternatively, the expression for equivalent plate thickness can be derived by evaluating the rigidities in a manner it

is done for corrugated sheets. Denoting the perimeter of the plate cross-section over the width b by s (following the loops of the stiffeners), we, therefore, represent rigidities as

$$\left.\begin{aligned} & D_y \simeq \frac{b}{s} D, \quad D_x \simeq \frac{I_s}{bh^3/12} D, \quad D_{xy} \simeq \frac{b}{s}(1-\nu)D \\ & \nu_x D_y \simeq \frac{b}{s}\nu D, \quad (\nu_x D_y + D_{xy}) = \frac{b}{s} D \end{aligned}\right\} \tag{2.18}$$

Use of (2.18) in (2.14) and a subsequent simplification yields

$$\frac{\bar{h}}{h} = \left[\frac{b}{2s} + \left(\frac{3I_s}{sh^3}\right)^{1/2} \right]^{1/3} \tag{2.19}$$

Equations (2.17) and (2.19) are suggested as replacements for Eq. (2.1) to determine the 'equivalent' plate thickness in light gauge design. Numerical calculations show that there is a substantial difference between the value of $\bar{h}$ determined from (2.1) and (2.17) or (2.19); Eq. (2.1) really overestimates the thickness of the equivalent plate.

CHAPTER III

POSTBUCKLING ANALYSIS OF ORTHOTROPIC PLATES-I

3.1 Introduction:

A number of studies are available in literature concerning postbuckling analysis of flat rectangular isotropic elastic plates, the most comprehensive being that of Stein (1959). Except the work of Prabhakara and Chia (1973), there is no investigation which deals with the postbuckling of orthotropic plates under compression. In this chapter, the postbuckling analysis of rectangular orthotropic plates of symmetric x-section is carried out. A method of successive approximation, originally applied by Alexeev (1956), is utilized. Von Karman's large deflection equations are converted into an infinite set of linear differential equations by expanding the displacements into a power series in terms of an arbitrary parameter. The first few equations of the infinite set turn out to be the small deflection equations. Solutions of these and succeeding equations permits a study of the behaviour of the plate at buckling and beyond, up into the large deflection range. The curves of load-shortening and effective width are presented for various cases of plates having different elastic properties. The possibility of a change

in the buckling pattern of the plate is also indicated.

3.2 Governing Equations:

For a plate with no lateral load, the von Karman large-deflection equations can be written in the form

$$N_{x,x} + N_{xy,y} = 0 \tag{3.1a}$$

$$N_{y,y} + N_{xy,x} = 0 \tag{3.1b}$$

$$\nabla w - (N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy}) = 0 \tag{3.1c}$$

where $\nabla w = D_x w_{,xxxx} + 2(\nu D + D_{xy}) w_{,xxyy} + D_y w_{,yyyy}$

Here a comma denotes partial differentiation with respect to the suffix that follows.

The strain-membrane force relations are

$$\epsilon_x = \frac{1}{h}\left(\frac{N_x}{E_x} - \nu_y \frac{N_y}{E_y}\right) \tag{3.2a}$$

$$\epsilon_y = \frac{1}{h}\left(\frac{N_y}{E_y} - \nu_x \frac{N_x}{E_x}\right) \tag{3.2b}$$

$$\gamma_{xy} = \frac{1}{h}\,\frac{N_{xy}}{G_{xy}} \tag{3.2c}$$

The forces in Eqs. (3.2) can be solved for and are expressed as

$$N_x = \frac{E_x h}{1-\nu_x\nu_y}(\epsilon_x + \nu_y \epsilon_y) \tag{3.3a}$$

$$N_y = \frac{E_y h}{1-\nu_x\nu_y}(\nu_x \epsilon_x + \epsilon_y) \tag{3.3b}$$

$$N_{xy} = G_{xy} h \gamma_{xy} \tag{3.3c}$$

The strain-displacement relations are

$$\epsilon_x = u_{,x} + \frac{1}{2} w^2_{,x} \tag{3.4a}$$

$$\epsilon_y = v_{,y} + \frac{1}{2} w^2_{,y} \tag{3.4b}$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y} \tag{3.4c}$$

It is assumed that u, v, and w may be expanded in a power series in terms of an arbitrary parameter α:

$$u = \alpha^o u_o + \alpha^2 u_2 + \alpha^4 u_4 + \text{-----} \tag{3.5a}$$

$$v = \alpha^o v_o + \alpha^2 v_2 + \alpha^4 v_4 + \text{-----} \tag{3.5b}$$

$$w = \alpha w_1 + \alpha^3 w_3 + \alpha^5 w_5 + \text{-----} \tag{3.5c}$$

For plates (without initial eccentricities) subject to in-plane loading, the deflection w is zero in the loading range prior to buckling but u and v have values other than zero. The series for u and v is therefore expected to start with zero power of α while the series for w is expected to start with a nonzero power. From the available solutions of the postbuckling behaviour of rectangular plates subjected to longitudinal compression, it is seen that the square of the centre deflection is nearly a linear function of the

applied load. For the compression problem it is convenient to relate α to the load P so that $\alpha^2 = (P-P_o)/P_o$, where P_o is the critical load. Thus, the series for w will start with the first power. The odd powers in the series of u and v and the even powers in the series of w vanish for problems of the type considered and, for simplicity, they have been omitted from the start.

It is advantageous to expand the externally applied loads also in terms of the arbitrary parameter α. Upon substitution of (3.5) into (3,3) and (3.4), the following relations are obtained:

$$N_x = \sum_{n=0,2}^{\infty} N_{xn}\alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_{xmn}\alpha^{m+n} \tag{3.6a}$$

$$N_y = \sum_{n=0,2}^{\infty} N_{yn}\alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_{ymn}\alpha^{m+n} \tag{3.6b}$$

$$N_{xy} = \sum_{n=0,2}^{\infty} N_{xyn}\alpha^n + \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} N_{xymn}\alpha^{m+n} \tag{3.6c}$$

where

$$N_{xn} = C_1(u_{n,x} + \nu_y v_{n,y}) \tag{3.6d}$$

$$N_{yn} = C_2(\nu_x u_{n,x} + v_{n,y}) \tag{3.6e}$$

$$N_{xyn} = C_3(u_{n,y} + v_{n,x}) \tag{3.6f}$$

$$N_{xmn} = \frac{C_1}{2}(w_{m,x}w_{n,x} + \nu_y w_{m,y} w_{n,y}) = N_{xnm} \qquad (3.6g)$$

$$N_{ymn} = \frac{C_2}{2}(\nu_x w_{m,x}w_{n,x} + w_{m,y}w_{n,y}) = N_{ynm} \qquad (3.6h)$$

$$N_{xymn} = C_3 \, w_{m,x}w_{n,y} \qquad (3.6l)$$

in which

$$C_1 = E_x h/(1-\nu_x\nu_y) \ , \quad C_2 = E_y h/(1-\nu_x\nu_y), \ C_3 = G_{xy}h$$

Since α is taken to be an arbitrary parameter, the stipulation that a power series in α vanish requires that each coefficient of power series vanish. Substituting (3.5c) and (3.6) in (3.1), this requirement leads to the following linear equations, which are the first few of an infinite set:

$$\left.\begin{aligned} N_{xo,x} + N_{xyo,y} &= 0 \\ N_{yo,y} + N_{xyo,x} &= 0 \end{aligned}\right\} \qquad (3.7a)$$

$$\nabla w_1 - (N_{xo} w_{1,xx} + N_{yo} w_{1,yy} + 2N_{xyo} w_{1,xy}) = 0 \qquad (3.7b)$$

$$\left.\begin{aligned} N_{x2,x} + N_{xy2,y} &= -(N_{x11,x} + N_{xy11,y}) \\ N_{y2,y} + N_{xy2,x} &= -(N_{y11,y} + N_{xy11,x}) \end{aligned}\right\} \qquad (3.7c)$$

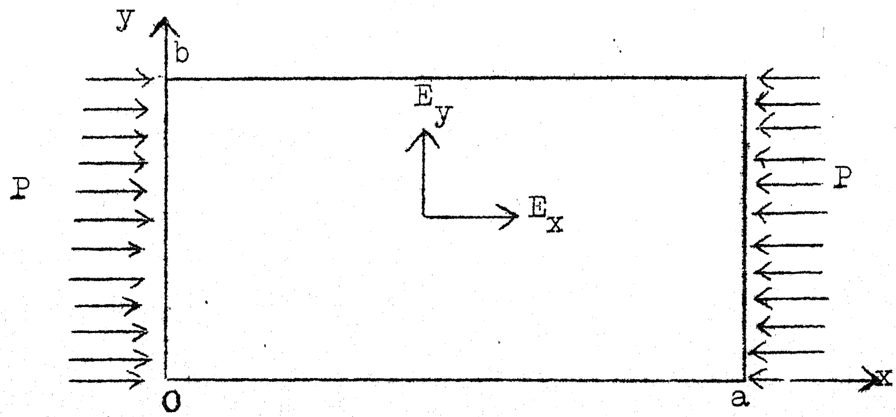
$$\begin{aligned} \nabla w_3 &- (N_{xo} w_{3,xx} + N_{yo} w_{3,yy} + 2N_{xyo} w_{3,xy}) \\ &= (N_{x2}+N_{x11}) w_{1,xx} + (N_{y2}+N_{y11}) w_{1,yy} \\ &\quad + 2(N_{xy2} + N_{xy11}) w_{1,xy} \end{aligned} \qquad (3.7d)$$

$$\left.\begin{aligned} N_{x4,x}+N_{xy4,y} &= -(2N_{x13,x}+N_{xy13,y}+N_{xy31,y}) \\ N_{y4,y}+N_{xy4,x} &= -(2N_{y13,y}+N_{xy13,x}+N_{xy31,x}) \end{aligned}\right\} \tag{3.7e}$$

$$\begin{aligned} \nabla w_5 - (N_{xo} w_{5,xx} + N_{yo} w_{5,yy} + 2N_{xyo}\, w_{5,xy}) \\ = (N_{x2}+N_{x11})\, w_{3,xx} + (N_{y2}+N_{y11})\, w_{3,yy} \\ +2(N_{xy2}+N_{xy11}) w_{3,xy} + (N_{x4}+2N_{x13}) w_{1,xx} \\ +(N_{y4}+2N_{y13}) w_{1,yy} + 2(N_{xy4}+N_{xy13}+N_{xy31}) w_{1,xy} \end{aligned} \tag{3.7f}$$

3.3 Solution Procedure:

The problem to be solved is the postbuckling behaviour of a simply supported plate under longitudinal compression with edges constrained so that the displacement of each edge in the plane of the plate is uniform. Accordingly, the following boundary conditions are imposed on the plate:



Zero deflection:

$$w(o,y) = w(a,y) = w(x,o) = w(x,b) = 0 \tag{3.8a}$$

Zero moment:

$$w_{,xx}(o,y) = w_{,xx}(a,y) = w_{,yy}(x,o) = w_{,yy}(x,b) = 0 \tag{3.8b}$$

Constant displacement:

$$u_{,y}(o,y) = u_{,y}(a,y) = v_{,x}(x,o) = v_{,x}(x,b) = 0 \tag{3.8c}$$

Zero shear stress:

$$v_{,x}(o,y) = v_{,x}(a,y) = u_{,y}(x,o) = u_{,y}(x,b) = 0 \tag{3.8d}$$

Loaded edges:

$$\int_o^b (N_x)_{x=o,a}\, dy = -P \tag{3.8e}$$

Unloaded edges:

$$\int_o^a (N_y)_{y=o,b}\, dx = 0 \tag{3.8f}$$

Substituting (3.6a,b) and noting that $P = P_o + \alpha^2 P_o$, the preceding force boundary conditions take the following form for different approximations in α :

Loaded edges:

$$\int_o^b (N_{xo})_{x=o,a}\, dy = -P_o \tag{3.9a}$$

$$\int_o^b (N_{x2} + N_{x11})_{x=o,a}\, dy = -P_o \tag{3.9b}$$

$$\int_o^b (N_{x4} + 2N_{x13})_{x=o,a}\, dy = 0 \tag{3.9c}$$

Unloaded edges:

$$\int_{o}^{a} (N_{yo})_{y=o,b}\, dx = 0 \tag{3.10a}$$

$$\int_{o}^{a} (N_{y2}+N_{y11})_{y=o,b}\, dx = 0 \tag{3.10b}$$

$$\int_{o}^{a} (N_{y4}+2N_{y13})_{y=o,b}\, dx = 0 \tag{3.10c}$$

3.3.1 <u>Fundamental Solution:</u>

Eqn. (3.7a) can be written in terms of the displacements u_o and v_o :

$$C_1 u_{o,xx}+C_3 u_{o,yy}+(\nu_y C_1+C_3)v_{o,xy} = 0$$

$$(\nu_x C_2+C_3)\, u_{o,xy}+C_2 v_{o,yy}+C_3 v_{o,xx} = 0$$

The solution of the above equations satisfying the boundary conditions (3.8c,d) and 3.9a) is

$$u_o = -\frac{P_o}{E_x bh}\left(x-\frac{a}{2}\right) \tag{3.11a}$$

$$v_o = \frac{P_o \nu_x}{E_x bh}\left(y-\frac{b}{2}\right) \tag{3.11b}$$

It therefore follows that

$$N_{xo} = -P_o/b\ ,\quad N_{yo} = N_{xyo} = 0 \tag{3.12}$$

Using (3.12), Eq. (3.7b) can now be written as

$$\nabla w_1 + \frac{P_o}{b} w_{1,xx} = 0 \qquad (3.13)$$

whose solution can be taken to be of the form

$$w_1 = A_1 \operatorname{Sin} Mx \operatorname{Sin} Ny; \; M = \frac{m\pi}{a}, \; N = \frac{n\pi}{b} \qquad (3.14)$$

which satisfies the boundary conditions (3.8a,b). This solution requires that

$$P_o = bM^{-2}[\, D_x M^4 + 2(\nu D + D_{xy})M^2 N^2 + D_y N^4 \,] \qquad (3.15)$$

So far, the solutions obtained are identical to those obtained from the small-deflection theory; P_o, therefore is to be identified as the buckling load. The lowest value of the buckling load is determined by the choice of m and n for a particular aspect ratio a/b. It may be noted that the amplitude A_1 can not as yet be determined.

3.3.2 <u>First Approximation:</u>

Eqs. (3.7c) are rewritten in terms of u_2, v_2, and w_1:

$$C_1 u_{2,xx} + C_3 u_{2,yy} + (\nu_y C_1 + C_3) v_{2,xy} = -C_1(w_{1,xx} w_{1,x} + \nu_y w_{1,xy} w_{1,y}) - C_3(w_{1,xy} w_{1,y} + w_{1,x} w_{1,yy})$$

$$(\nu_x C_2 + C_3) u_{2,xy} + C_2 v_{2,yy} + C_3 v_{2,xx}$$

$$= -C_2(w_{1,yy} w_{1,y} + \nu_x w_{1,xy} w_{1,x}) - C_3(w_{1,xy} w_{1,x} + w_{1,y} w_{1,xx})$$

Upon substitution for w_1 from (3.14) into preceding equations, the following solutions for u_2 and v_2 are obtained.

$$u_2 = -\left[\frac{P_o}{E_x bh} + \frac{A_1^2}{8} M^2\right] \left(x - \frac{a}{2}\right) - \frac{A_1^2}{16}\left[\frac{M^2 - \nu_y N^2}{M} \text{Sin } 2Mx - M \text{ Sin } 2Mx \text{ Cos } 2Ny\right] \quad (3.16a)$$

$$v_2 = \left[\frac{P_o \nu_x}{E_x bh} - \frac{A_1^2}{8} N^2\right] \left(y - \frac{b}{2}\right) - \frac{A_1^2}{16}\left[\frac{N^2 - \nu_x M^2}{N} \text{Sin } 2Ny - N \text{ Cos } 2Mx \text{ Sin } 2Ny\right] \quad (3.16b)$$

So that

$$\left.\begin{aligned} N_{x2} + N_{x11} &= -\frac{P_o}{b} - \frac{E_x h\, A_1^2}{8} M^2 \text{ Cos } 2\, Ny \\ N_{y2} + N_{y11} &= -\frac{E_x h\, A_1^2}{8} N^2 \text{ Cos } 2Mx \\ N_{xy2} + N_{xy11} &= 0 \end{aligned}\right\} \quad (3.17)$$

Now w_3 must be determined from Eq. (3.7d). After substitution of the N's and w_1, (3.7d) becomes

$$\nabla w_3 + \frac{P_o}{b} w_{3,xx} = A_1\left[\frac{P_o}{b} M^2 - \frac{A_1^2}{16} h(E_x M^4 + E_y N^4)\right] \text{Sin } Mx \text{ Sin } Ny + \frac{E_x h\, A_1^3}{16} M^4 \text{ Sin } Mx \text{ Sin } 3Ny + \frac{E_y h\, A_1^3}{16} N^4 \text{ Sin } 3Mx \text{ Sin } Ny \quad (3.18)$$

It should be noted that Sin Mx Sin Ny is a complementary solution of Eq. (3.18), and a term of this kind appears on the right hand side of this equation. No solution to

Eq. (3.18) is possible that satisfies boundary conditions of this problem unless the coefficient of this term on the right hand side of this equation is zero. Thus,

$$[\frac{P_o}{b} M^2 - \frac{A_1^2}{16} h(E_x M^4 + E_y N^4)]\ A_1 = 0$$

This relationship provides the means of determining the value of A_1. If the trivial case $A_1 = 0$ is ignored, the amplitude A_1 is found to be

$$A_1^2 = \frac{16P_o}{bh} M^2\ (E_x M^4 + E_y N^4)^{-1} \tag{3.19}$$

Now the complete solution for w_3 which satisfies (3.18) and the boundary conditions can be written as:

$$w_3 = A_3 \text{Sin } Mx \text{ Sin } Ny + A_{13} \text{Sin } Mx \text{ Sin } 3Ny + A_{31} \text{ Sin } 3Mx \text{Sin } Ny \tag{3.20}$$

where A_3 can not be determined as yet, and

$$A_{13} = \frac{E_x h\ A_1^3}{16} M^4 [\ D_x M^4 + 2(\ \nu D + D_{xy}) M^2 (3N)^2 + D_y (3N)^4 - \frac{P_o}{b} M^2\ ]^{-1}$$

$$A_{31} = \frac{E_y h\ A_1^3}{16} N^4\ [D_x (3M)^4 + 2(\nu D + D_{xy})(3M)^2 N^2 + D_y N^4 - \frac{P_o}{b}(3M)^2]\ ^{-1}$$

3.3.3 <u>Second Approximation:</u>

First the expressions for N_{13}'s and N_{31}'s can be written in terms of w_3. Then, Eqs. (3.7e) become

$$C_1u_{4,xx}+C_3u_{4,yy}+(\nu_yC_1+C_3)v_{4,xy} = -C_1(w_{1,xx}w_{3,x}+w_{1,x}w_{3,xx})$$

$$-(\nu_yC_1+C_3)\ (w_{1,xy}w_{3,y}+w_{1,y}w_{3,xy})-C_3(w_{1,x}w_{3,yy}+w_{3,x}w_{1,yy})$$

$$(\nu_xC_2+C_3)u_{4,xy}+C_3v_{4,xx}+C_2v_{4,yy}$$

$$=-C_2(w_{1,yy}w_{3,y}+w_{1,y}w_{3,yy})-(\nu_xC_2+C_3)\ .$$

$$(w_{1,xy}w_{3,x}+w_{1,x}w_{3,xy})-C_3(w_{1,y}w_{3,xx}+w_{3,y}w_{1,xx})$$

The solutions that satisfy the boundary conditions are

$$u_4 = -\frac{A_1A_3M^2}{4}(x-\frac{a}{2}) - \frac{A_1}{8M}[A_3(M^2-\nu_yN^2)+A_{31}(3M^2+\nu_yN^2)]\ .$$

$$\text{Sin } 2Mx - \frac{A_1A_{31}}{16M}(3M^2-\nu_yN^2)\text{ Sin } 4Mx$$

$$+\frac{A_1M}{8}[A_3-A_{13}+3A_{31}-4C_3N^2\ \frac{(C_2N^2-\nu_yC_1M^2)(A_{13}+A_{31})}{C_1C_3M^4+C_2C_3N^4+\bar{C}M^2N^2}]$$

$$\text{Sin } 2Mx\text{Cos } 2Ny + \frac{A_1A_{13}M}{8}[1+C_3N^2\ \frac{(4C_2N^2-\nu_yC_1M^2)}{C_1C_3M^4+C_2C_3(2N)^4+\bar{C}M^2(2N)^2}]$$

$$\text{Sin } 2Mx\text{ Cos } 4Ny+\frac{A_1A_{31}M}{4}[1-C_3M^2\ .$$

$$\frac{(4C_1M^2+C_1C_2(1-\nu_x\nu_y)N^2/C_3-\nu_xC_2N^2)}{C_1C_3(2M)^4+C_2C_3N^4+\bar{C}(2M)^2N^2}]\text{ Sin } 4Mx\text{ Cos } 2Ny \tag{3.21a}$$

where $\bar{C} = C_1C_2(1-\nu_x\nu_y) - C_1C_3\nu_y-C_2C_3\nu_x$

$$v_4=-\frac{A_1A_3N^2}{4}(y-\frac{b}{2})-\frac{A_1}{8N}[A_3(N^2-\nu_xM^2)+A_{13}(3N^2+\nu_xM^2)]\text{ Sin } 2Ny$$

$$-\frac{A_1A_{13}}{16N}(3N^2-\nu_xM^2)\text{Sin}4Ny+\frac{A_1N}{8}[A_3-A_{31}+3A_{13}$$

$$-4C_3M^2\ \frac{(C_1M^2-\nu_xC_2N^2)(A_{13}+A_{31})}{C_1C_3M^4+C_2C_3N^4+\bar{C}M^2N^2}]\text{Cos}2Mx\text{ Sin } 2Ny$$

$$+\frac{A_1A_{13}N}{4}\left[1-C_3N^2\frac{(4C_2N^2+C_1C_2(1-\nu_x\nu_y)M^2/C_3-\nu_yC_1M^2)}{C_1C_3M^4+C_2C_3(2N)^4+\bar{C}M^2(2N)^2}\right]\text{Cos}2Mx\,\text{Sin}4Ny$$

$$+\frac{A_1A_{31}N}{8}\left[1+C_3M^2\frac{(4C_1M^2-\nu_xC_2N^2)}{C_1C_3(2M)^4+C_2C_3N^4+\bar{C}(2M)^2N^2}\right]\text{Cos}4Mx\,\text{Sin}2Ny$$

(3.21b)

so that

$$N_{x4}+2N_{x13}=-\frac{A_1M^2E_{xh}}{4}\left[(A_3-A_{13})\text{Cos}2Ny+A_{13}\text{Cos}\,4Ny\right.$$

$$+4C_2C_3N^4(A_{13}+A_{31})\{C_1C_3M^4+C_2C_3N^4+\bar{C}M^2N^2\}^{-1}\text{Cos}2Mx\text{Cos}2Ny$$

$$-4C_2C_3N^4A_{13}\{C_1C_3M^4+C_2C_3(2N)^4+\bar{C}M^2(2N)^2\}^{-1}\text{Cos}2Mx\text{Cos}4Ny$$

$$\left.-C_2C_3N^4A_{31}\{C_1C_3(2M)^4+C_2C_3N^4+\bar{C}(2M)^2N^2\}^{-1}\text{Cos}4Mx\text{Cos}2Ny\right]$$

(3.22a)

$$N_{y4}+2N_{y13}=-\frac{A_1N^2E_yh}{4}\left[(A_3-A_{31})\text{Cos}2Mx+A_{31}\text{Cos}4Mx\right.$$

$$+4C_1C_3M^4(A_{13}+A_{31})\{C_1C_3M^4+C_2C_3N^4+\bar{C}M^2N^2\}^{-1}\text{Cos}2Mx\text{Cos}2Ny$$

$$-C_1C_3M^4A_{13}\{C_1C_3M^4+C_2C_3(2N)^4+\bar{C}M^2(2N)^2\}^{-1}\text{Cos}2Mx\text{Cos}4Ny$$

$$\left.-4C_1C_3M^4A_{31}\{C_1C_3(2M)^4+C_2C_3(N)^4+\bar{C}(2M)^2N^2\}^{-1}\text{Cos}4Mx\text{Cos}2Ny\right]$$

(3.22b)

$$N_{xy4}+N_{xy13}+N_{xy31}=-\frac{A_1M^3N^3}{2}C_3(1-\nu_x\nu_y)\left[2C_1C_2(A_{13}+A_{31})\right.$$

$$\{C_1C_2M^4+C_2C_3+\bar{C}M^2N^2\}^{-1}\text{Sin}2Mx\text{Sin}\,2Ny$$

$$-C_1C_2A_{13}\{C_1C_3M^4+C_2C_3(2N)^4+\bar{C}M^2(2N)^2\}^{-1}\text{Sin}2Mx\,\text{Sin}4Ny$$

$$\left.-C_1C_2A_{31}\{C_1C_3(2M)^4+C_2C_3N^4+\bar{C}(2M)^2N^2\}^{-1}\text{Sin}4Mx\,\text{Sin}2Ny\right]$$

(3.22c)

The differential Eq. (3.7f) for w_5 is now completely determined except for A_3. The constant A_3 is determined in the same way in which A_1 was determined; thus

$$A_3 = [E_x A_{13} M^4 + E_y A_{31} N^4]\,[E_x M^4 + E_y N^4 - \frac{16 P_o M^2}{3bh\, A_1^2}]^{-1} \tag{3.23}$$

Therefore, upto second approximateion, the displacement functions are

$$\left.\begin{aligned} u &= \alpha^o u_o + \alpha^2 u_2 + \alpha^4 u_4 \\ v &= \alpha^o v_o + \alpha^2 v_2 + \alpha^4 v_4 \\ w &= \alpha w_1 + \alpha^3 w_3 \end{aligned}\right\} \tag{3.24}$$

with $\alpha^2 = (P - P_o)/P_o$

3.4 Main Results:

Some of the important results can now be written down:

Critical Load: Using the expression in (2.4) for various rigidities, the formula (3.15) for critical load can be written as

$$P_o = \frac{\pi^2 D_x}{b} \cdot K$$

where

$$K = (\frac{mb}{a})^2 + 2\nu_y + \frac{4(1-\nu_x \nu_y)}{E_x/E_y + 1 + 2\nu_x} + (\frac{a}{mb})^2 \frac{E_y}{E_x} \tag{3.25}$$

is the nondimensionalized buckling coefficient.

<u>Load-shortening relationship:</u> Total axial shortening Δ is the sum of the inward displacements at each end. Since u is positive in the positive x-direction,

$$\Delta = u(o,y) - u(a,y) \qquad (3.26)$$

Substituting the expressions for u_o, u_2, and u_4 in the first of (3.24) and then using (3.26) will yield the following relation after lot of simplifications:

$$\frac{\Delta}{\Delta_o} = \left\{ \frac{P}{P_o} + \left(\frac{P}{P_o} - 1\right) \frac{2}{1+\beta} + \left(\frac{P}{P_o} - 1\right)^2 \left[\frac{3(1+2K_1+\beta)}{4(1+\beta)^3} \cdot \left(\frac{1}{2(K_1+5\beta)} + \frac{\beta^2}{9-\beta} \right) \right\} \right] \qquad (3.27)$$

where Δ_o is the axial shortening corresponding to critical load, i.e., $\Delta_o = P_o a / E_x bh$

and $\beta = \frac{E_y}{E_x} \left(\frac{a}{mb}\right)^4$

$$K_1 = \nu_y + \frac{2(1-\nu_y \nu_y)}{1+E_x/E_y+2\nu_x}$$

<u>Effective-width relation:</u> The effective width can conveniently be defined as

$$\frac{\Delta}{a} = \frac{P}{h\, b_e\, E_x} \qquad (3.28)$$

Substituting Eq. (3.26) in the above expression, we get

$$\frac{b_e}{b} = \frac{P}{P_o}\left[\frac{P}{P_o} + 2\left(\frac{P}{P_o} - 1\right)/(1+\beta) + \left(\frac{P}{P_o} - 1\right)^2 \{3(1+2K_1+\beta)/\right.$$

$$\left. 4(1+\beta)^3\{1/2(K_1+5\beta)+\beta^2/(9-\beta)\}\}\right]^{-1} \qquad (3.29)$$

3.5 Discussions:

Numerical calculations for the results are performed for various plates having different elastic properties defined by the parameters E_x, E_y, ν_x, ν_y. However, for the purpose of presentation here, the following three cases are considered:

(i) E_x/E_y (to be denoted, henceforth, by E^*) = 1 , $\nu_x = \nu_y = \nu = 0.3$. It corresponds to an isotropic plate.

(ii) $E^* = 4$, $\nu_x = 0.3$, $\nu_y = 0.075$

(iii) $E^* = 0.5$, $\nu_x = 0.15$, $\nu_y = 0.3$.

In Figs. 3-1 to 3-3, the critical stress coefficient K is plotted for three cases mentioned above. The portions of the curves defining the lowest critical values of the load are shown by full lines. The different a/b ratios at which the transition from m to (m+1) half-waves takes place for a particular value of E^* can be read of from the corresponding curve. For a given aspect ratio, the half-wave at the instant of buckling are more for $E^* < 1$, as

compared with the isotropic case, and are less for $E^* > 1$. For example, an isotropic plate of a/b ratio 3 will buckle in 3 half-waves whereas plates with $E^* = 4$ and 0.5 of the same ratio will buckle in 2 and 4 half-waves, respectively. Therefore, the value of the parameter $\lambda (=mb/a)$ in these cases are 1, 2/3, 4/3. In Fig. 3-4, the variation of K is plotted for various values of λ; it monotonically increases with λ for $E^* \geq 1$ whereas for $E^* < 1$ it first decreases and then again increases as λ is increased.

Figs. 3-5 to 3-7 represent load shortening curves for a/b = 1.0, 1.5 , 3.0 respectively. For each a/b ratio, all the three value of E^* are taken. Load-shortening curves are drawn using the relation (3.27) in which the values of m correspond to the lowest buckling load; n always equals unity for this problem. In addition, the curves are also given for other values of m which intersect with these basic curves for the range plotted. There are similar to those obtained by Stein (1959) for isotropic plates. The intersection of the load-shortening curves indicate possible changes in buckling pattern. For finite plates, changes in buckling pattern are often observed experimentally. However, experiment by Stein (1959) shows that the change does not necessarily occur at the intersection points of these lines for different numbers of halfwave.

Load-shortening curves also give an idea of the postbuckling strength of the plate which depends upon the ratio E^* and the parameter λ. Fig. 3-8 (a-c) show such curves for $\lambda = 1.0$, 1.5, 2.0 respectively. For $\lambda = 1$, all the plates show distinct postbuckling strength; plates with $E^* < 1$ have a higher strength compared to plates with $E^* > 1$. The situation is just the reverse for $\lambda = 2$. For $\lambda = 1.5$ and 2, the postbuckling strength of plate with $E^* = 0.5$ is not much different than that of the isotropic plate.

Fig. 3-9 is a plot of non-dimensional effective width (b_e/b) versus λ for $P/P_o = 2$, based on Eq.(3.29). Figs. 3-10(a-c), on the other hand, give the variation of b_e/b with nondimensional load P/P_o for $\lambda=1.0$, 1.5, 2.0, respectively. In Fig. 3-10(a) the dotted curves correspond to the results based on the first approximation only. If a/b is given, the designer can determine the appropriate value of m from load-shortening curves, for a selected value of P/P_o and the known properties of the plate. This determines λ, which is mb/a. Once λ is known, the corresponding value of the effective width can be found out from Fig. 3-9 or 3-10.

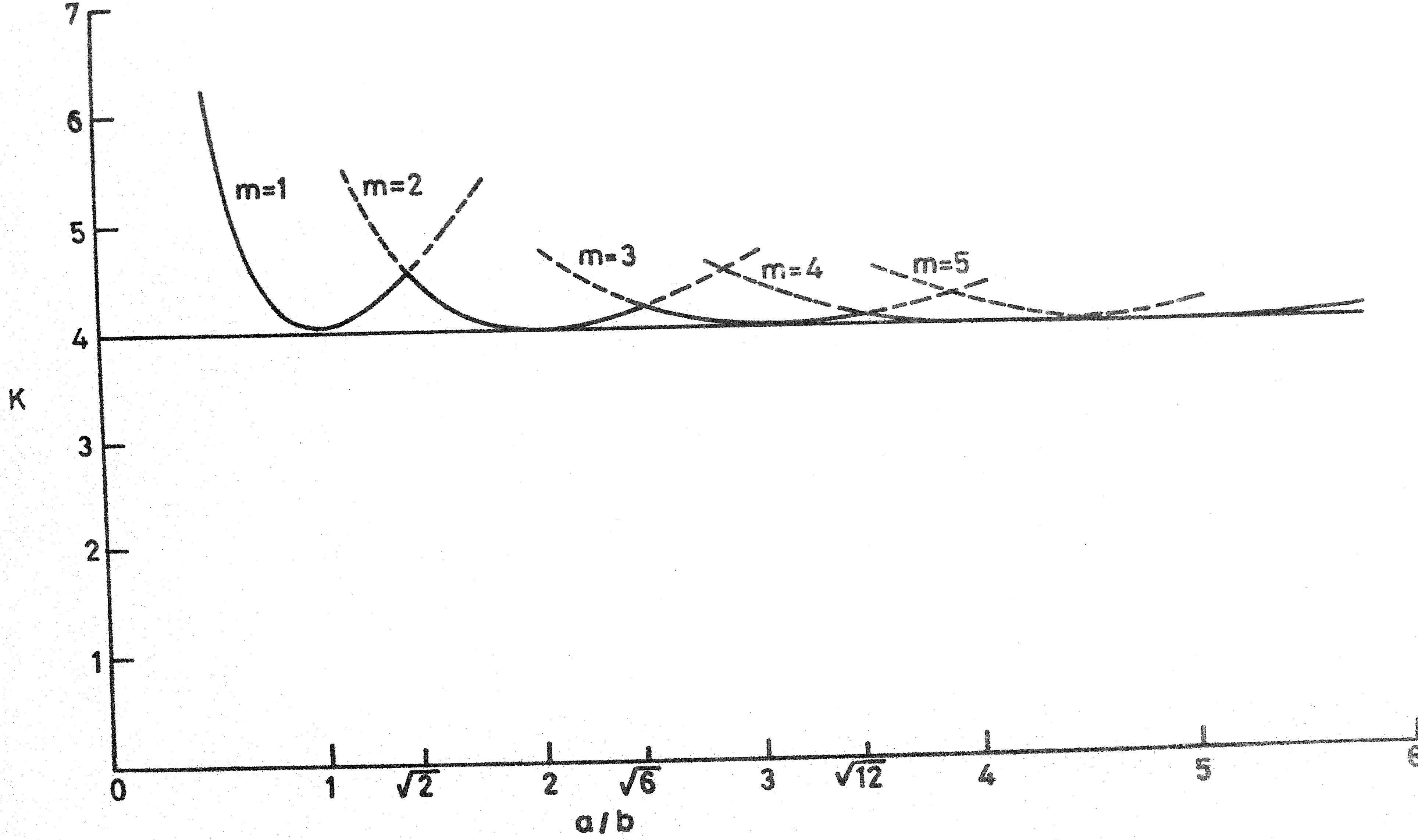


Fig.3-1 Buckling stress coefficient K (Ex / Ey=1)

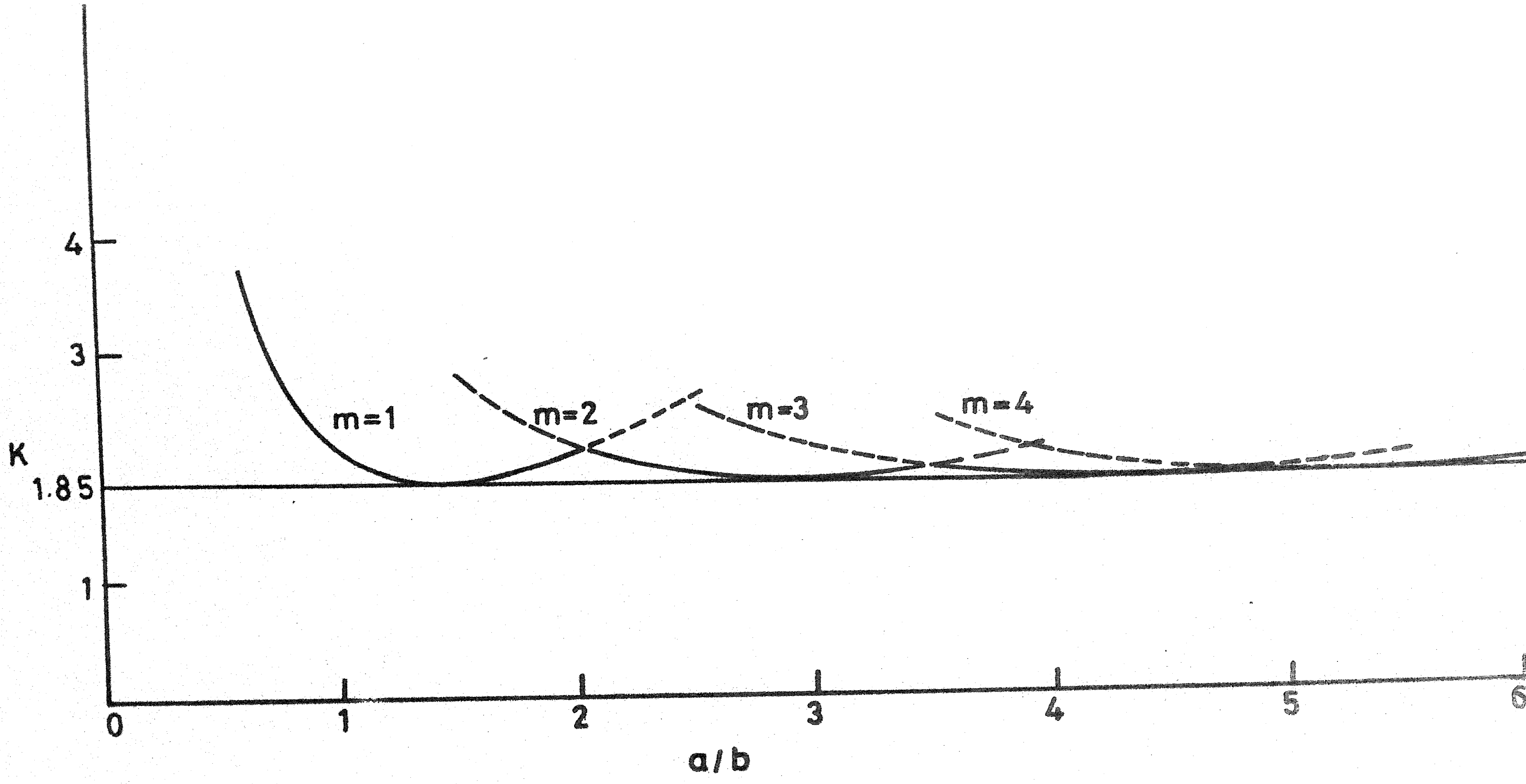


Fig. 3-2 Buckling stress coefficient K (Ex / Ey=4)

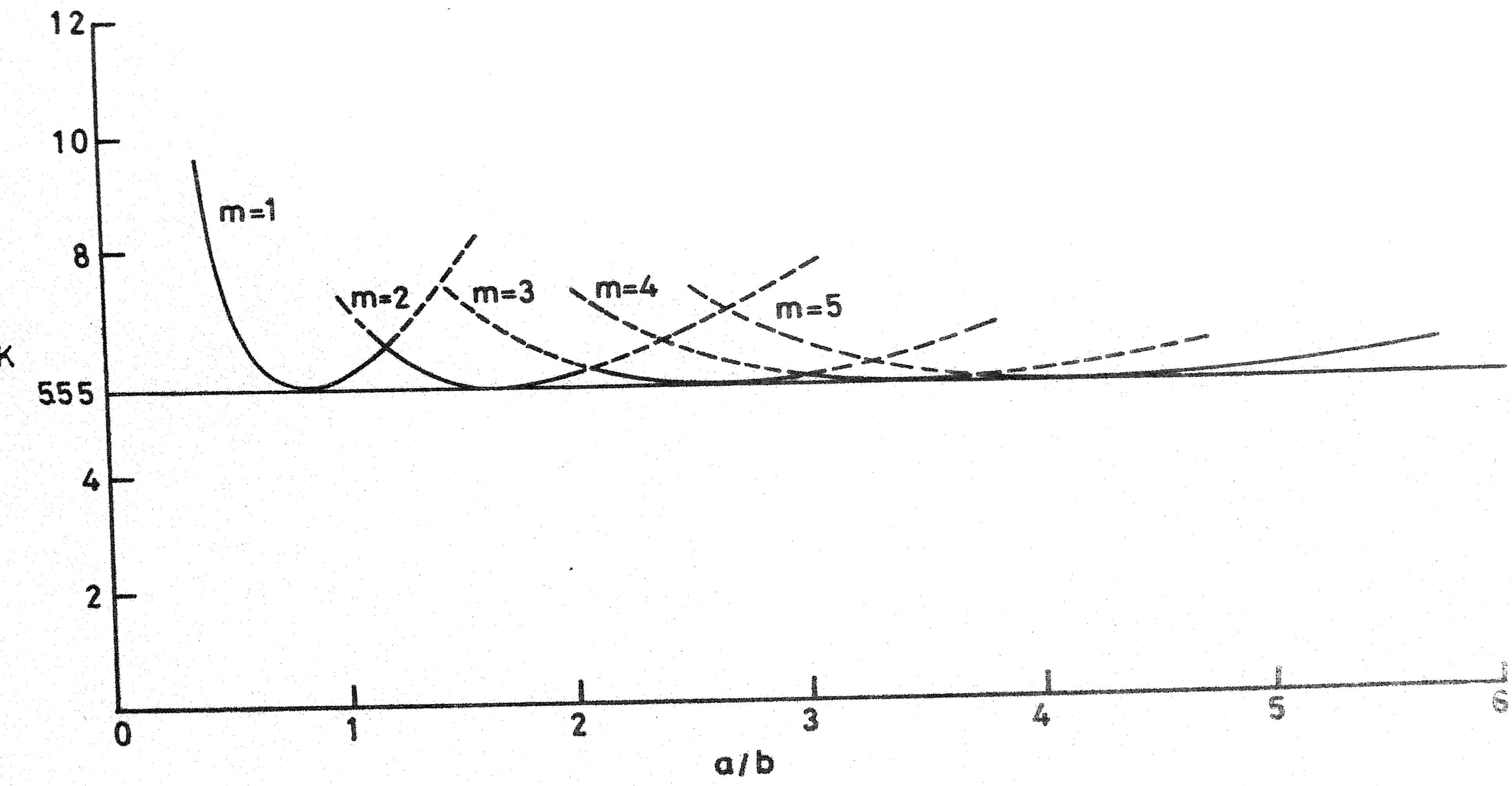


Fig.3-3 Buckling stress coefficient K (Ex /Ey = 0.5)

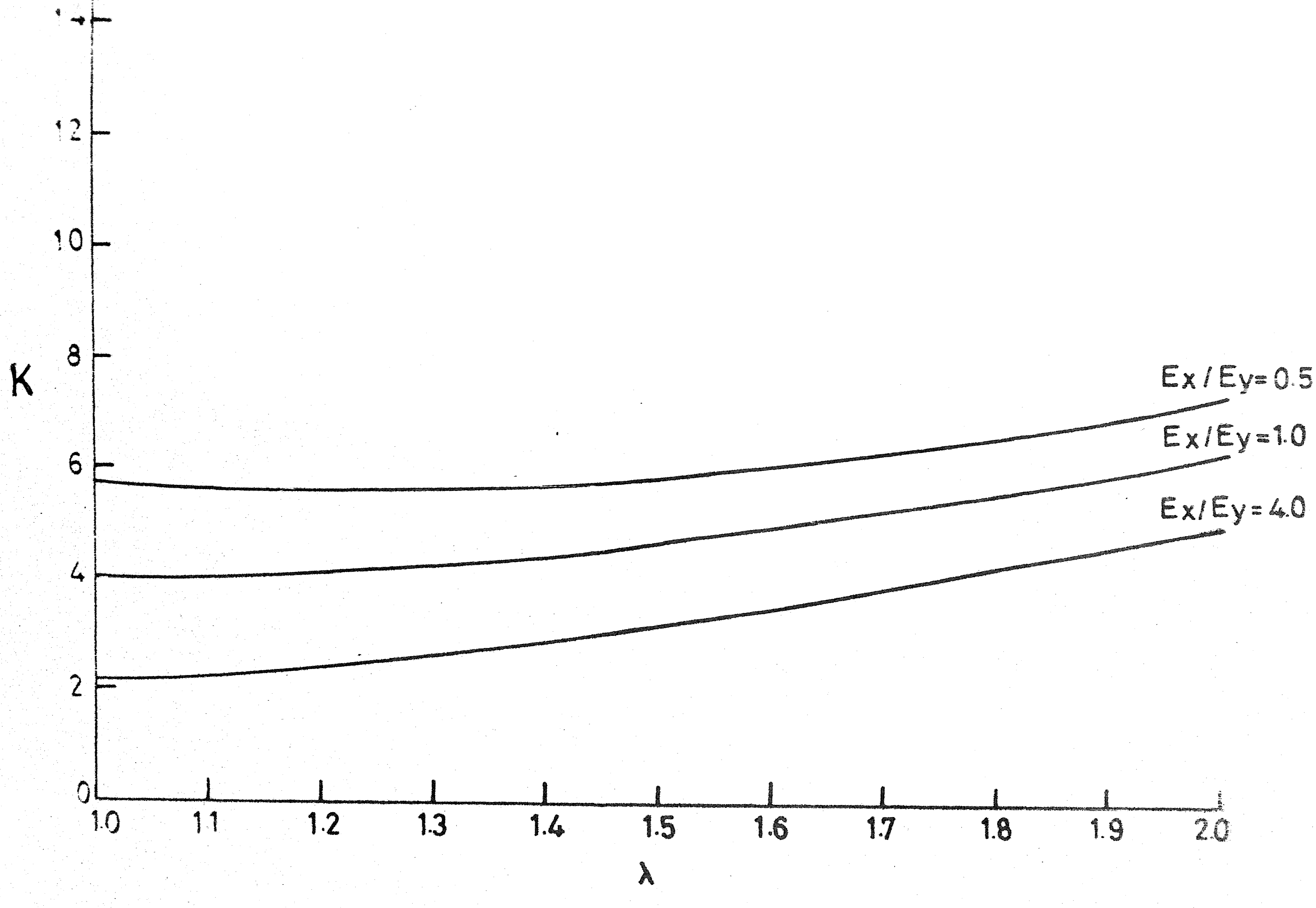


Fig. 3-4 Buckling stress coefficient K for different λ.

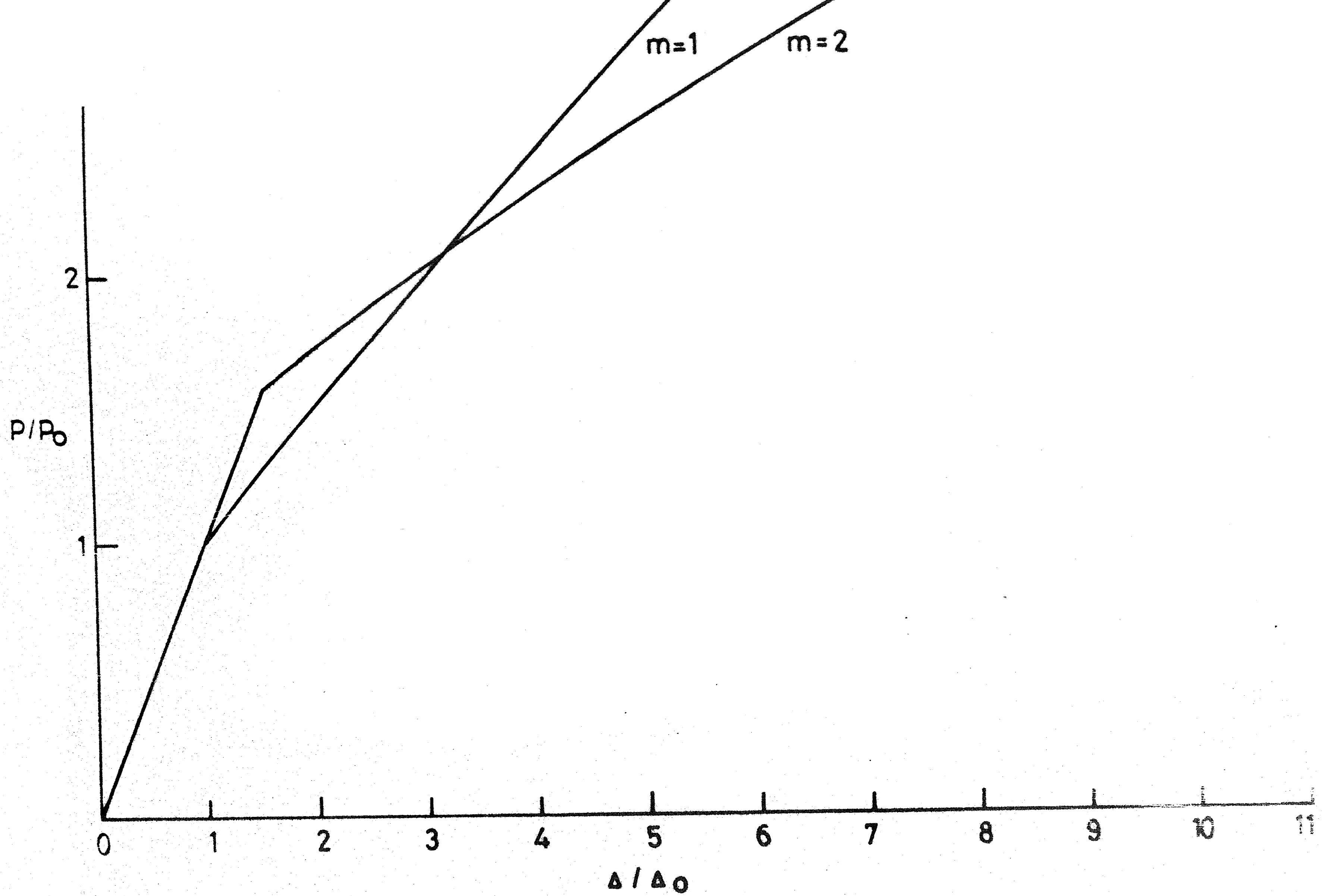


Fig. 3-5(a) Nondimensional load shortening curve (a/b=1 & Ex / Ey=1)

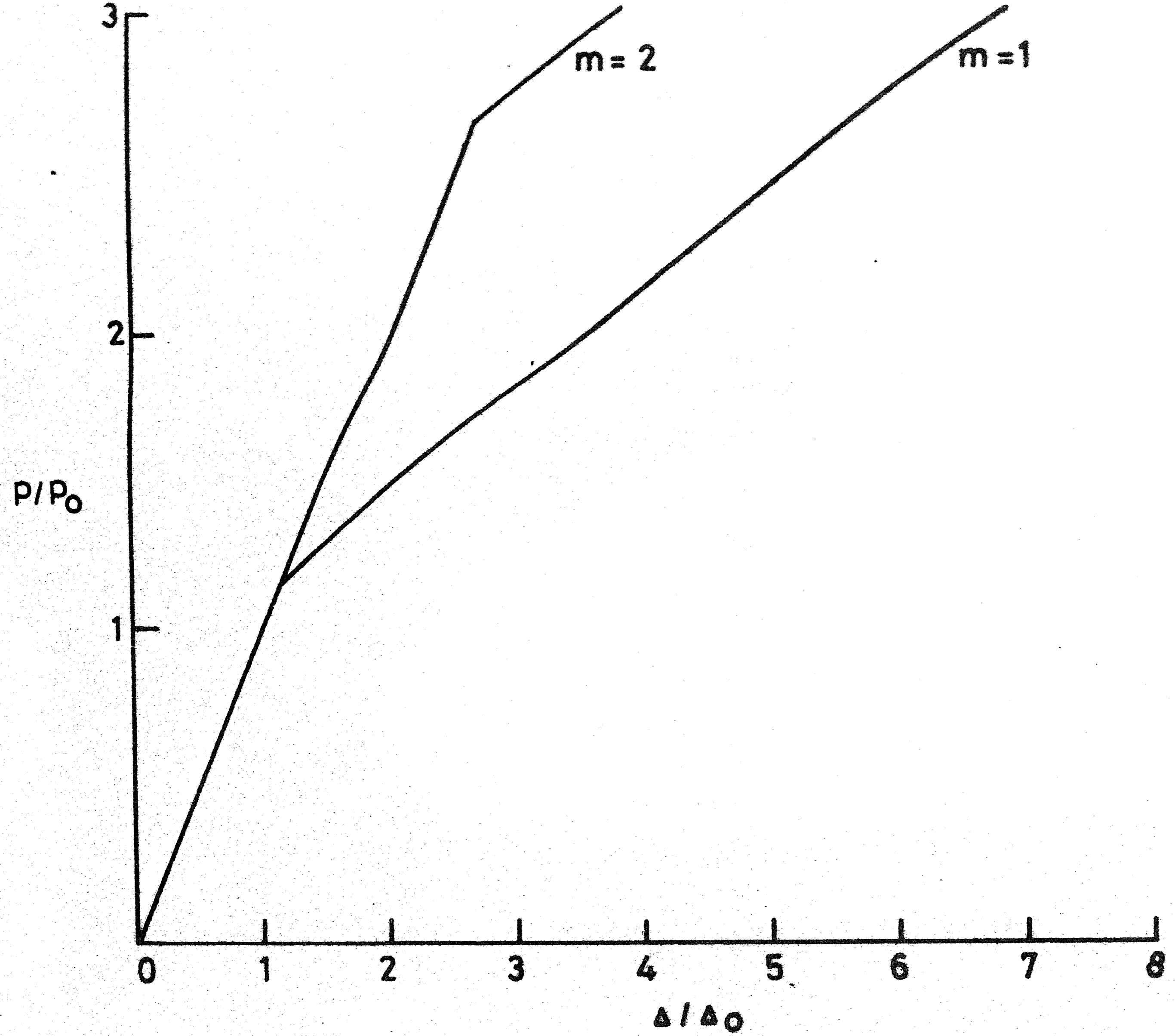


Fig. 3-5(b) Nondimensional load shortening curve
(a/b =1 & Ex/Ey = 4)

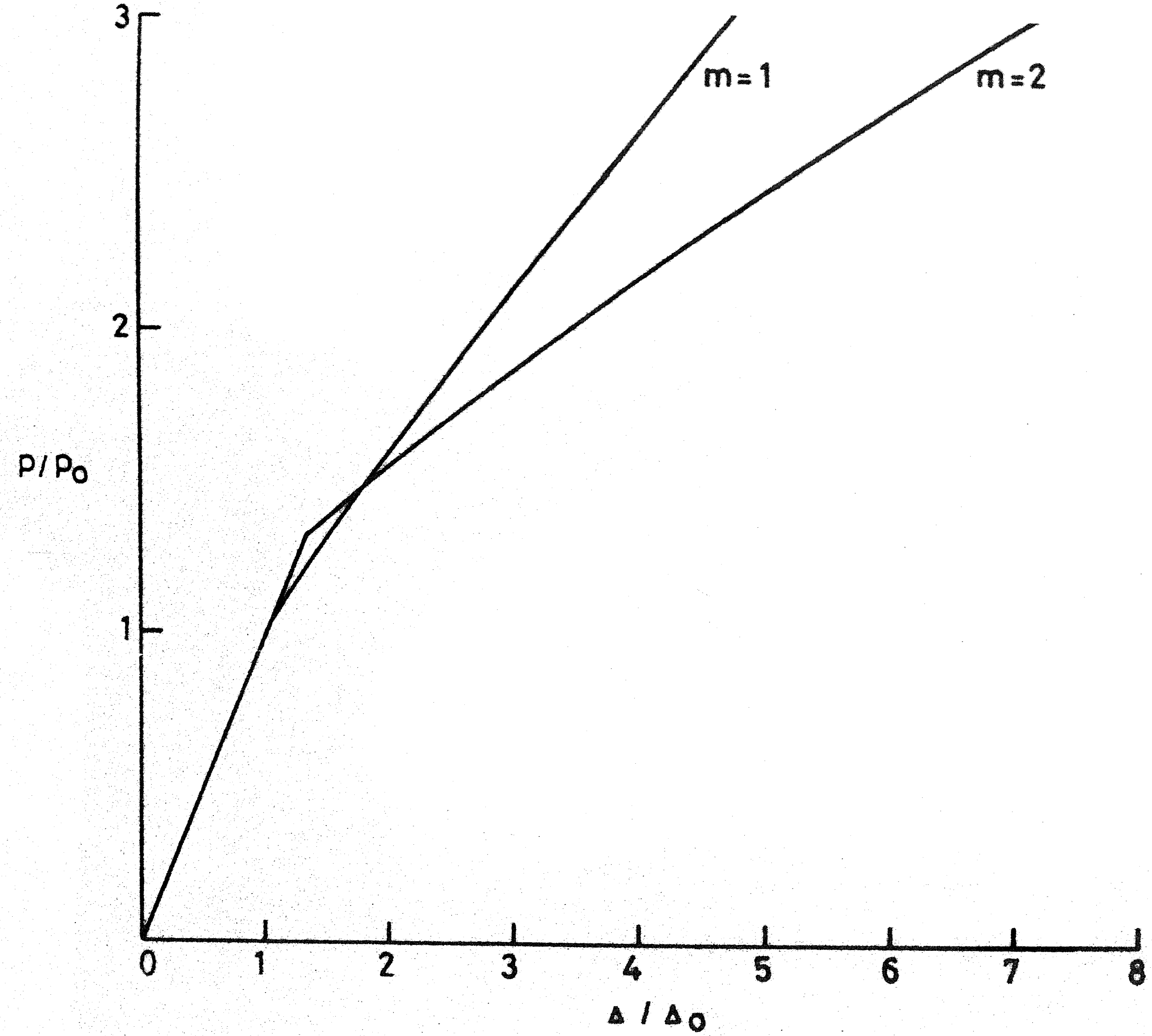
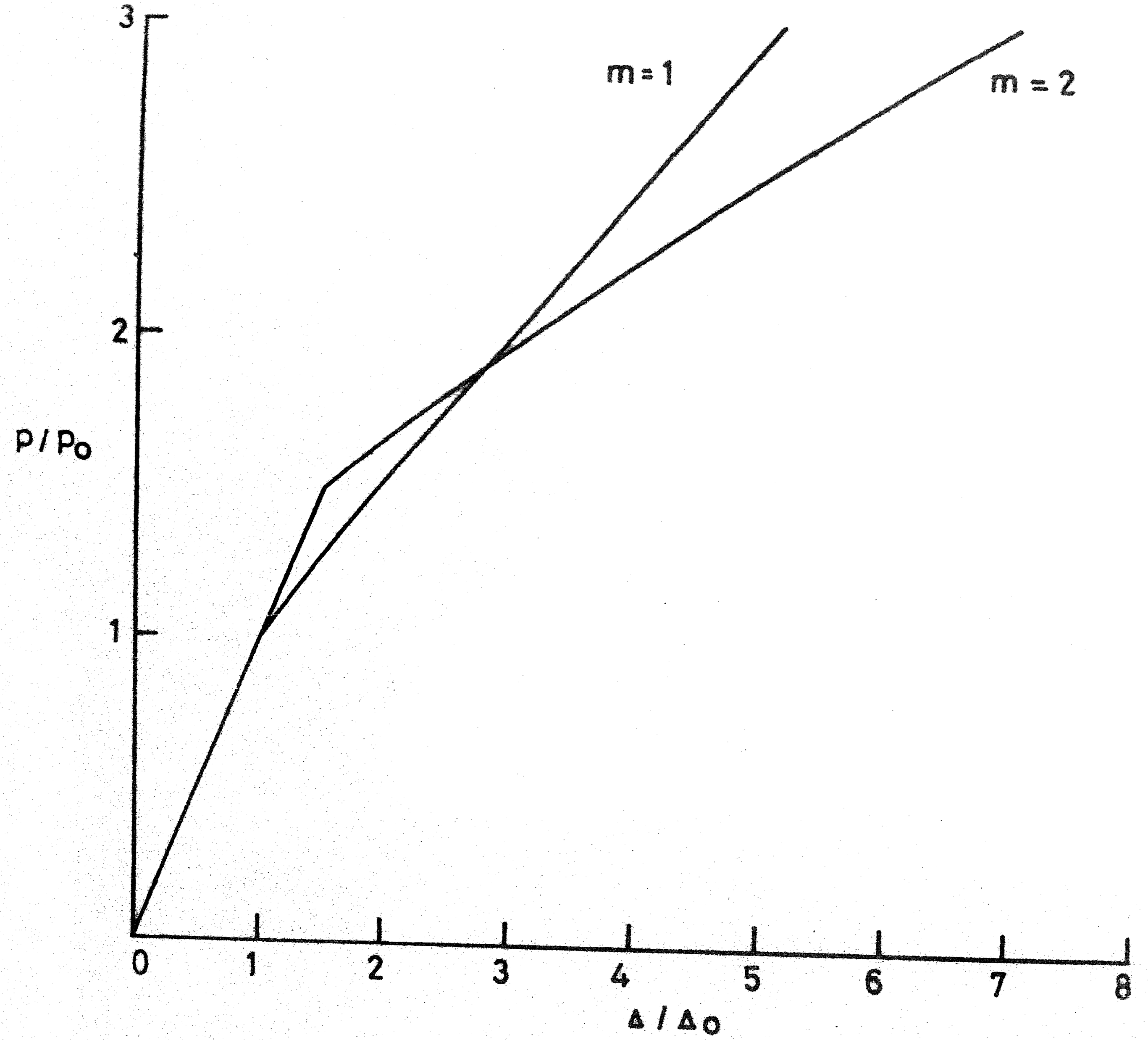


Fig. 3-5(c) Nondimensional load shortening curve
($a/b=1$ & $E_x / E_y = 0.5$)


m = 1
m = 2
P / Po
Δ / Δo
0
1
2
3
4
5
6
7
8
1
2
3

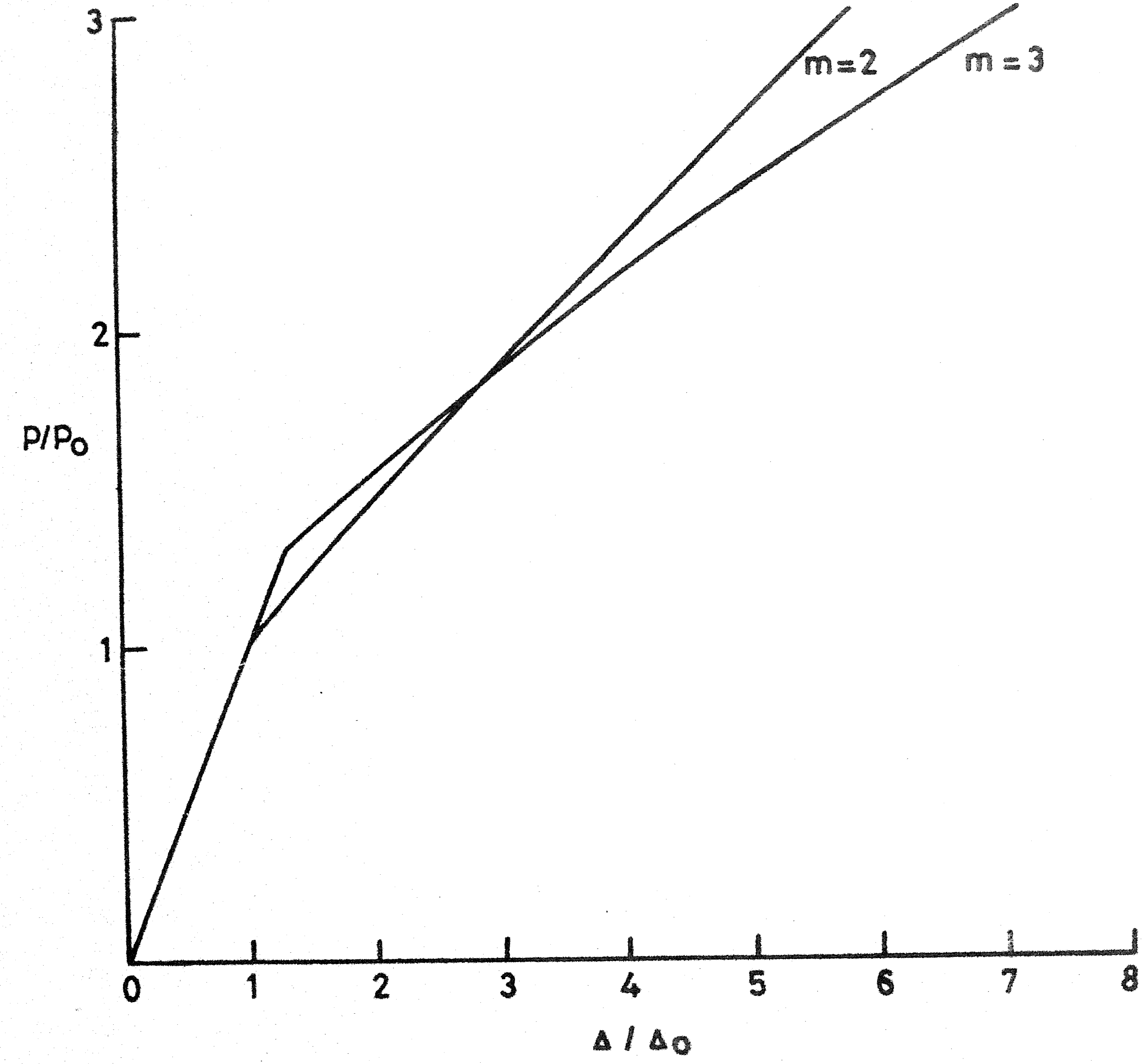
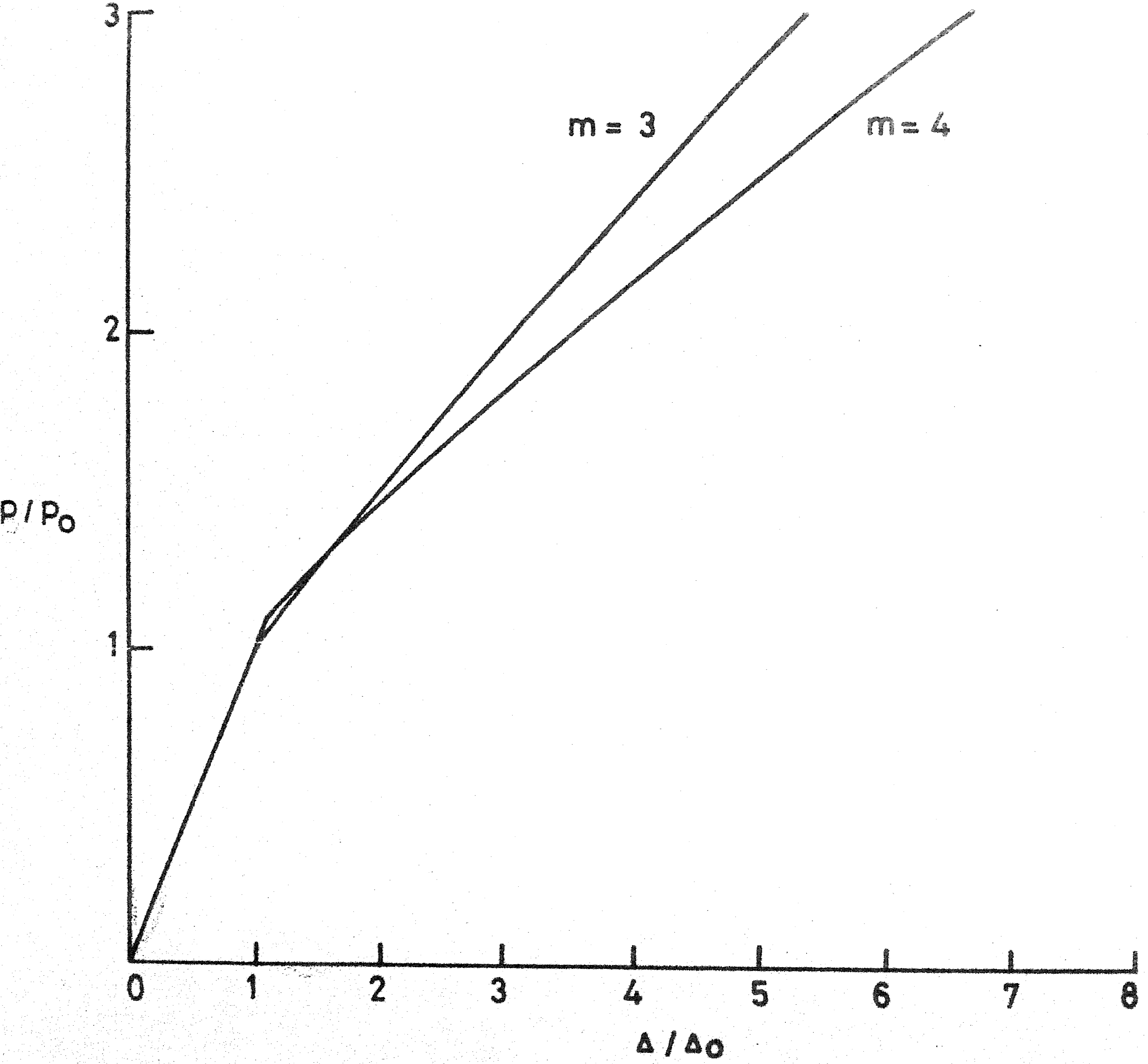
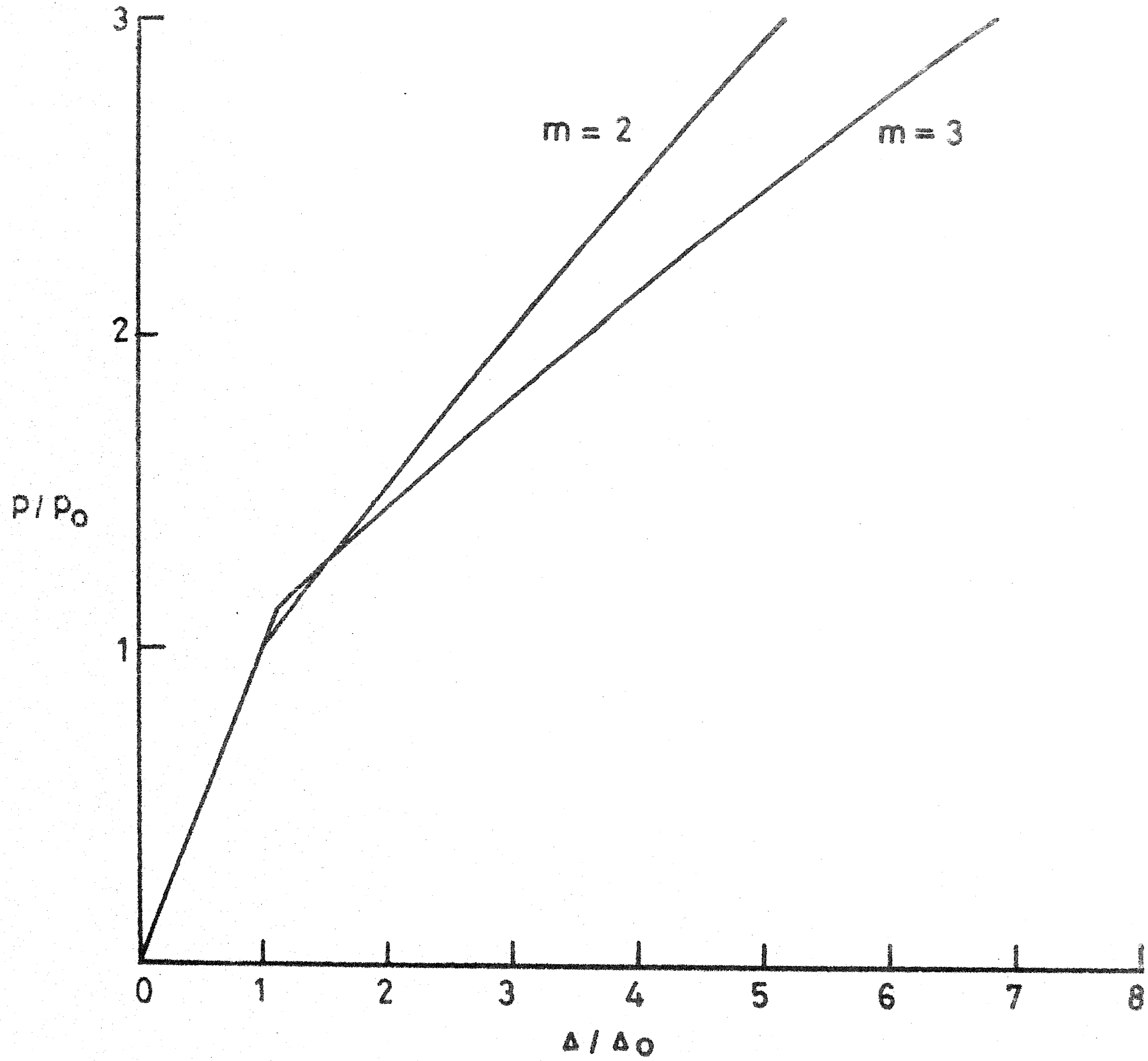


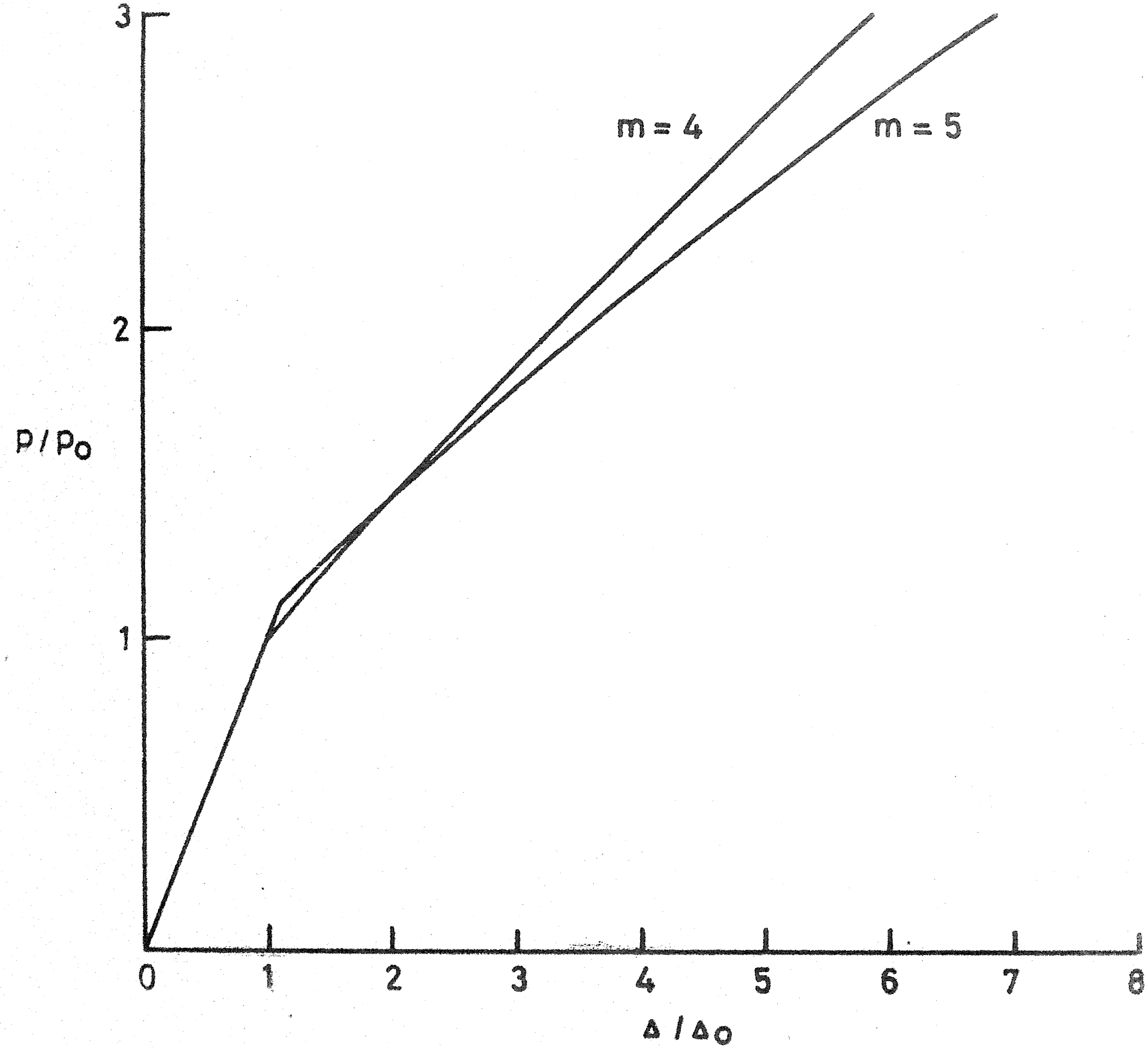
Fig. 3-6(c) Nondimensional load shortening curve
($a/b = 1.5$ & $E_x/E_y = 0.5$)

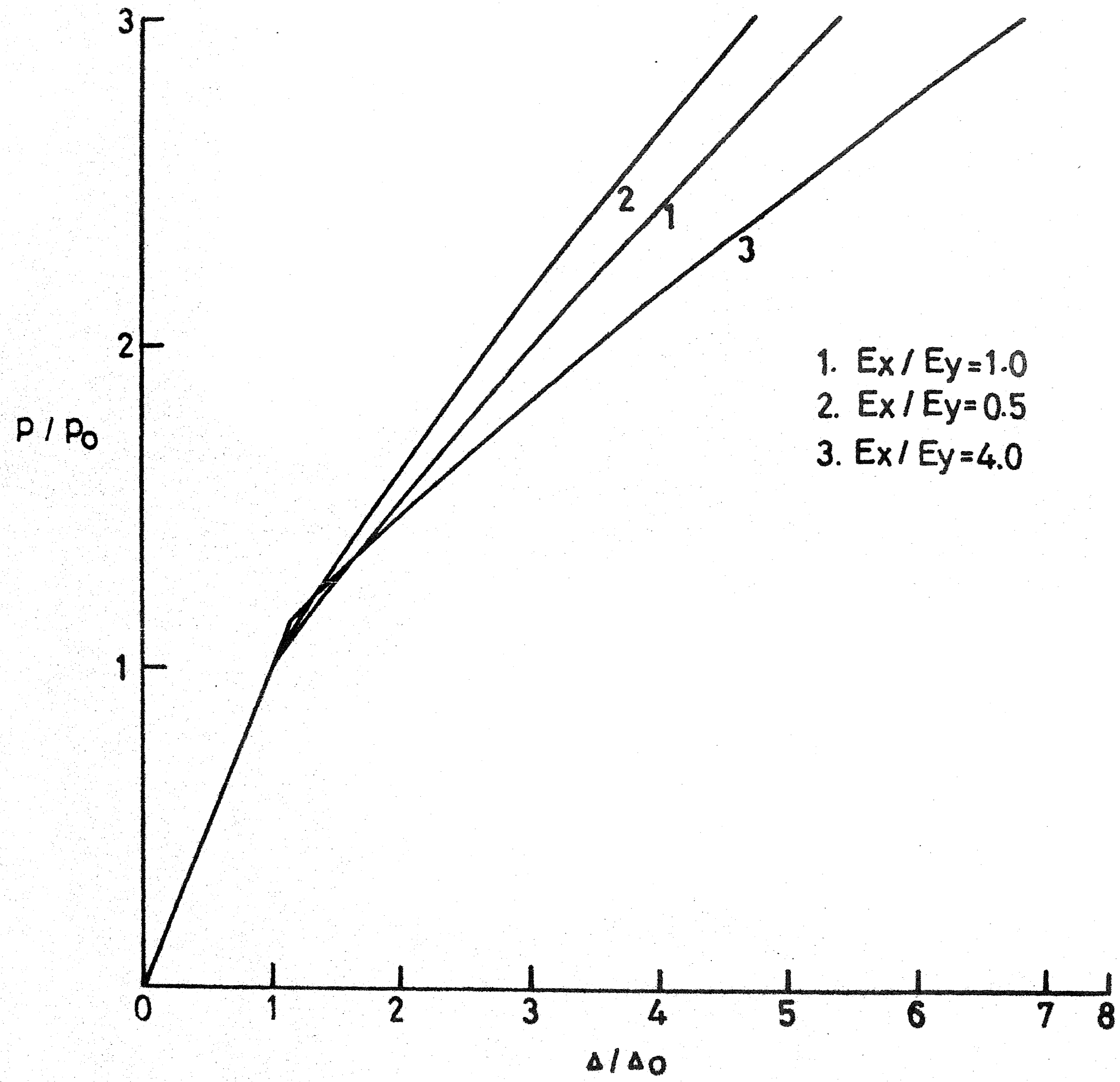


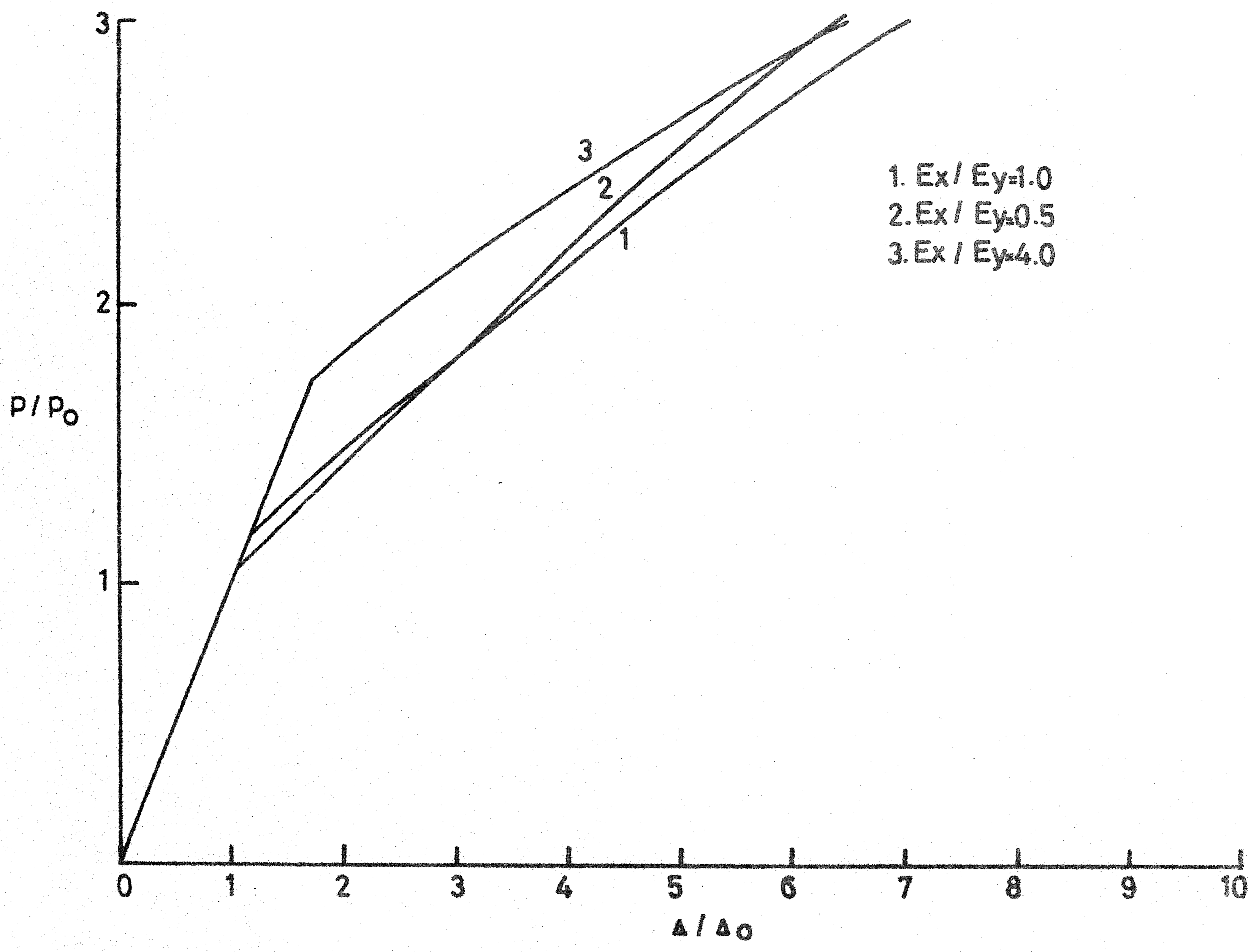
P / Po
m = 3
m = 4
0
1
2
3
4
5
6
7
8
Δ / Δo



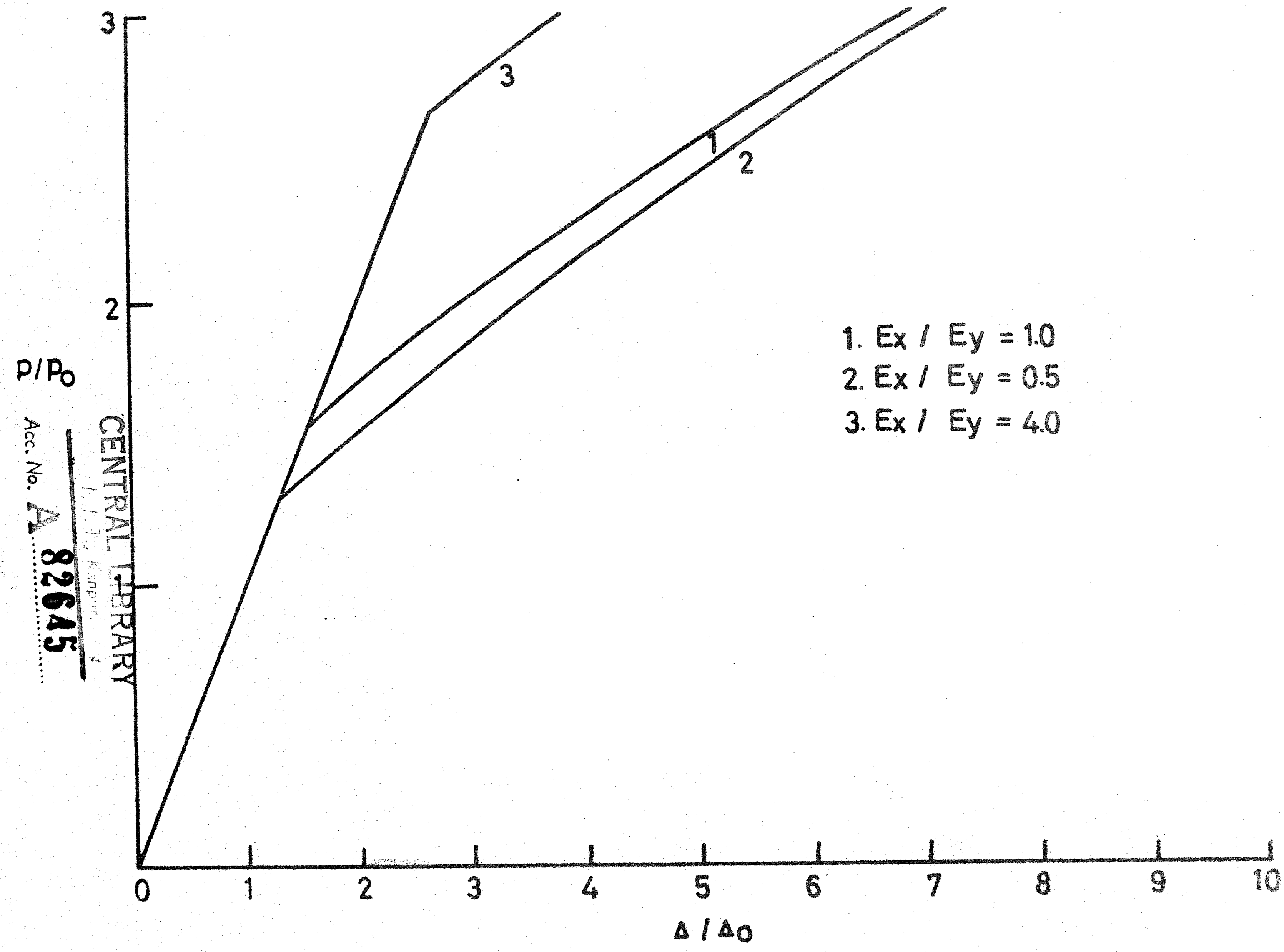
3
2
1
P / Po
m = 2
m = 3
0
1
2
3
4
5
6
7
8
Δ / Δo

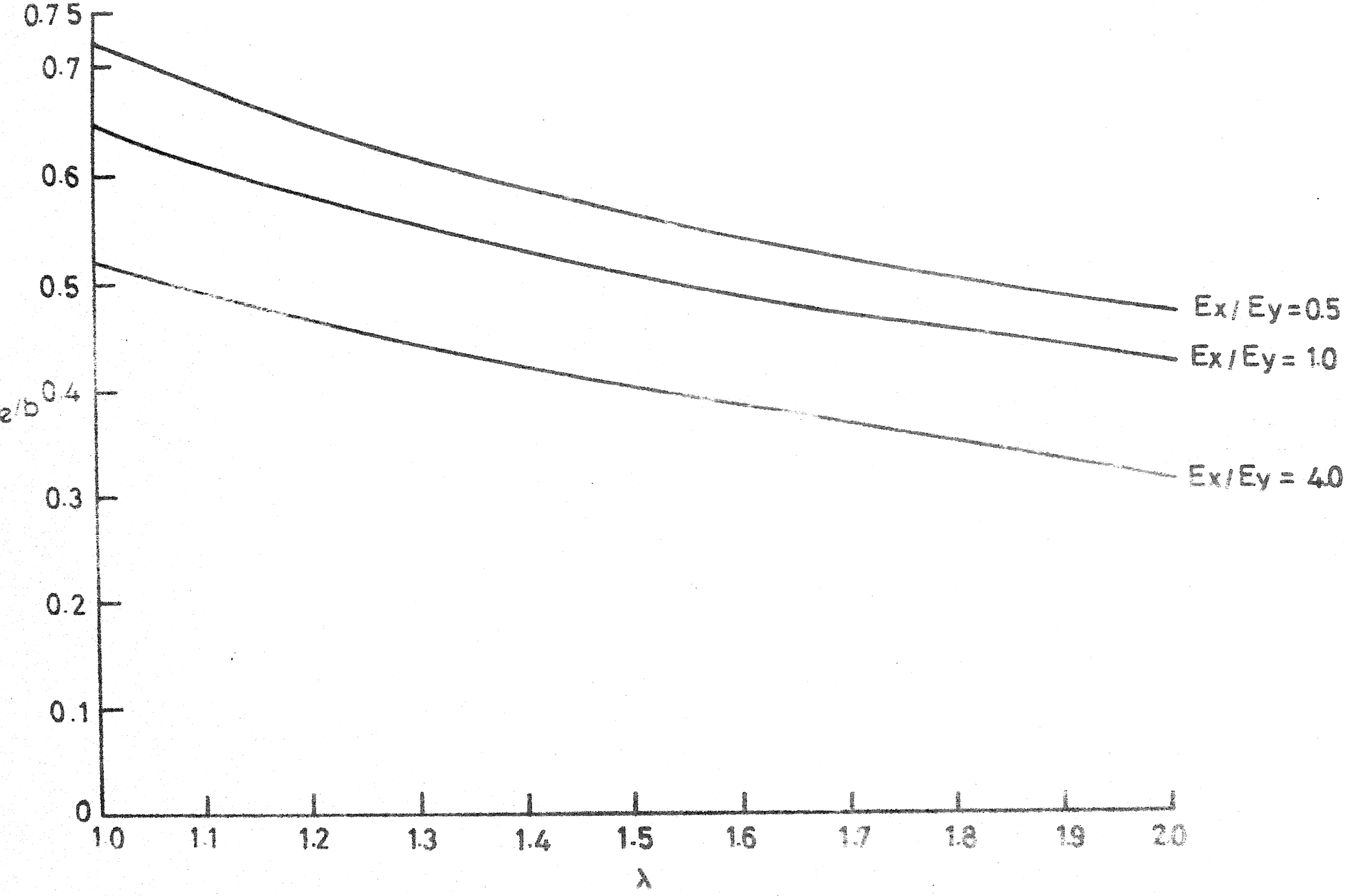

m = 4
m = 5
P / Po
Δ / Δo
0
1
2
3
4
5
6
7
8


P / Po
Δ / Δo
0
1
2
3
4
5
6
7
8
1. Ex / Ey = 1.0
2. Ex / Ey = 0.5
3. Ex / Ey = 4.0

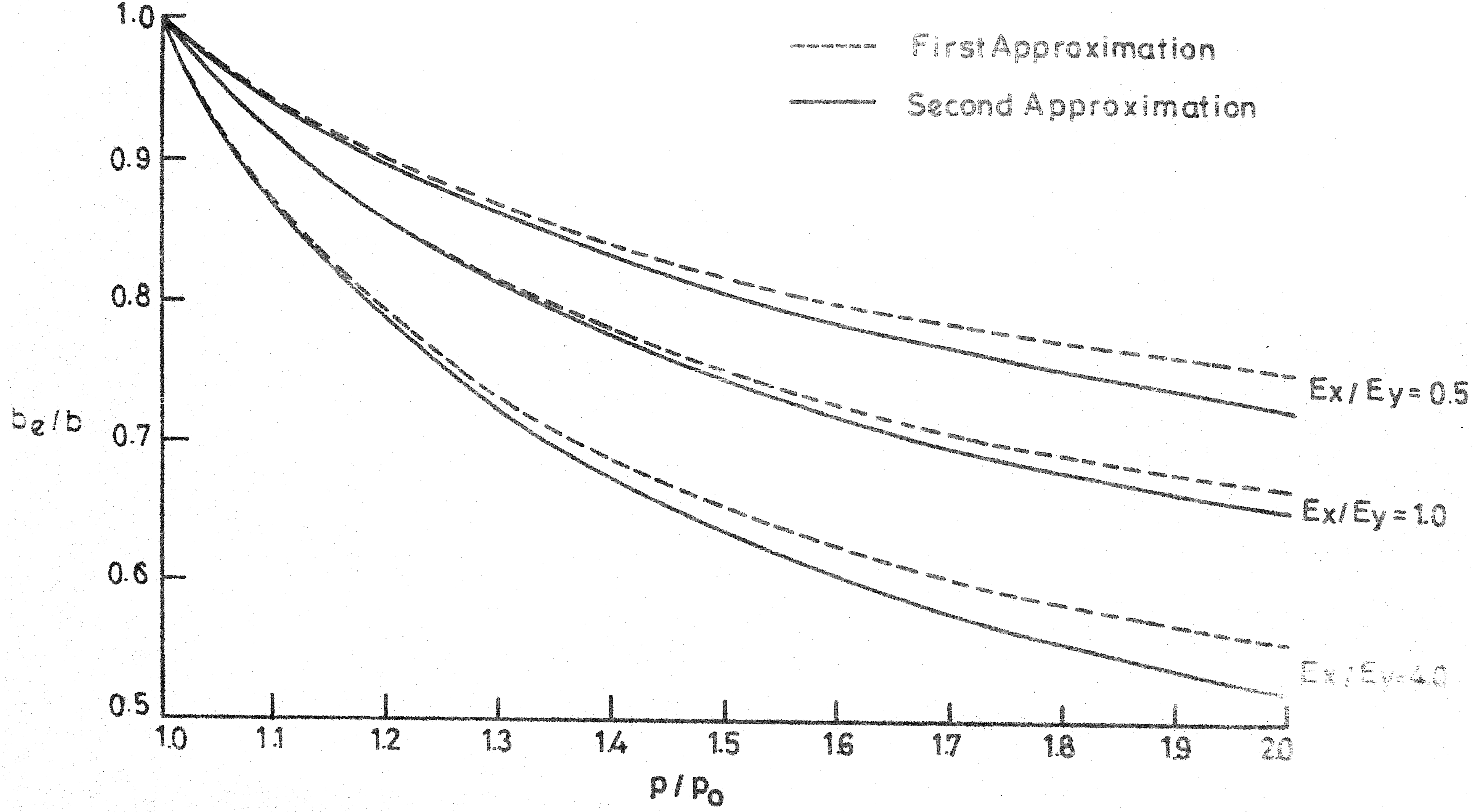


3
2
1
0
P / Po
1 2 3 4 5 6 7 8 9 10
Δ / Δo
1. Ex / Ey=1.0
2. Ex / Ey=0.5
3. Ex / Ey=4.0

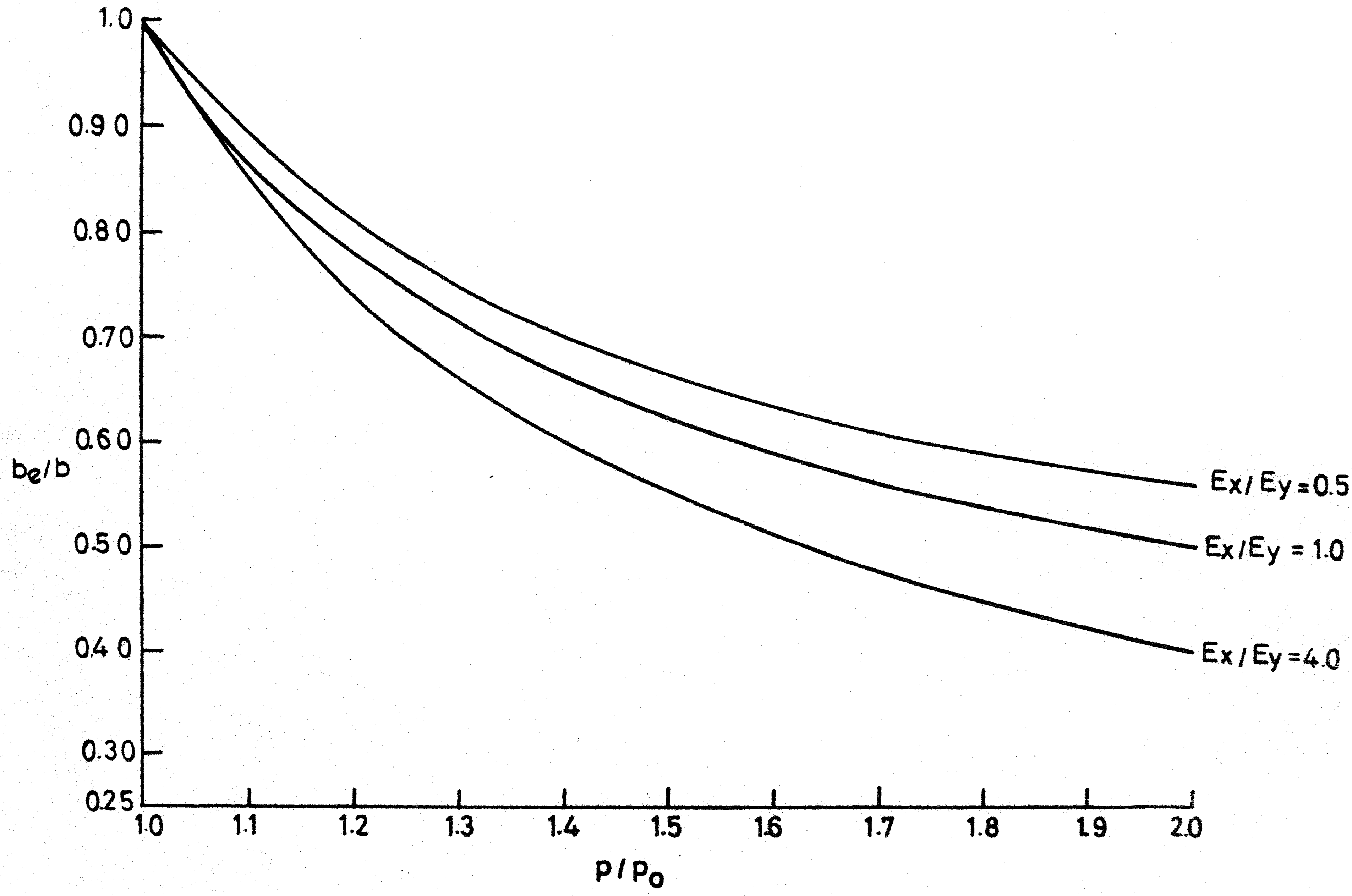

3
2
P/Po
0
1
2
3
4
5
6
7
8
9
10
Δ / Δo
1
2
3
1. Ex / Ey = 1.0
2. Ex / Ey = 0.5
3. Ex / Ey = 4.0

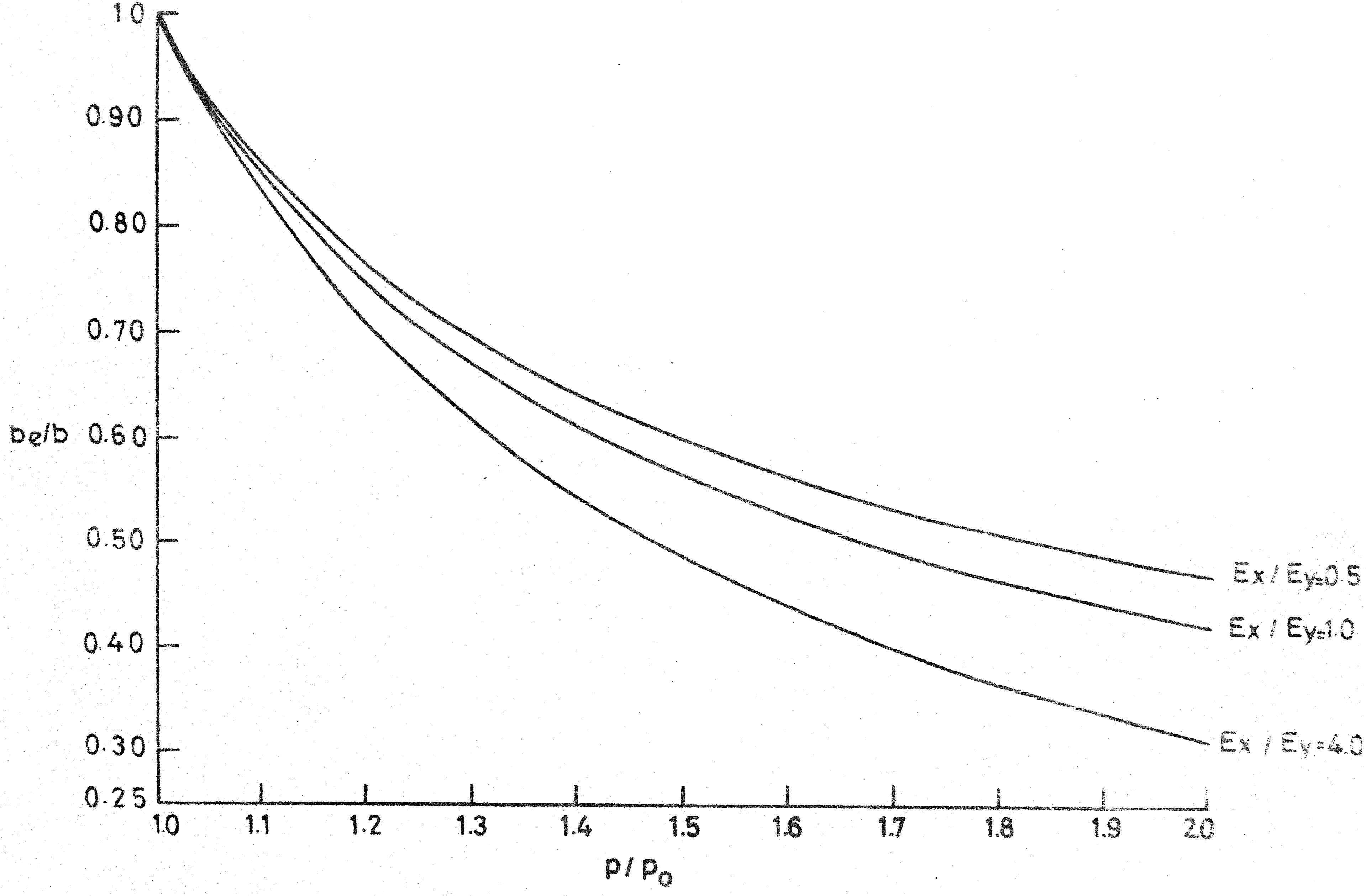


0.75
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
λ
Ex / Ey = 0.5
Ex / Ey = 1.0
Ex / Ey = 4.0



First Approximation
Second Approximation
Ex / Ey = 0.5
Ex/Ey = 1.0
Ex / Ey = 4.0
b_e / b
1.0
0.9
0.8
0.7
0.6
0.5
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
P / P_o


1.0
0.9 0
0.8 0
0.70
0.6 0
0.5 0
0.4 0
0.30
0.25
be/b
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
p/po
Ex/Ey=0.5
Ex/Ey = 1.0
Ex/Ey=4.0


1.0
0.90
0.80
0.70
be/b 0.60
0.50
0.40
0.30
0.25
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
p / po
Ex / Ey=0.5
Ex / Ey=1.0
Ex / Ey=4.0

CHAPTER IV

POSTBUCKLING ANALYSIS OF ORTHOTROPIC PLATES-II

4.1 Introduction:

A fairly rigorous analysis of the postbuckling strength of orthotropic plates has been carried out in Chapter 3. The aim in this chapter is to present an approximate analysis of the same problem. Results are somewhat less accurate to the extent that they match with the results obtained from the 'first approximation' only in Chapter 3. The method consists in assuming a simple function for w (the mid surface displacement), solving for w and a stress function F(x,y) and then evaluating the arbitrary constant in w by means of the Galerkin's method. A similar approach for isotropic plate can be seen in Chajes (1974).

4.2 Governing Relations:

As stated in Chapter 3, the equations of equilibrium are

$$\left.\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0 \end{aligned}\right\} \tag{4.1a}$$

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(\nu D + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - N_x \frac{\partial^2 w}{\partial x^2}$$

$$-N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0 \qquad (4.1b)$$

The middle-surface strains are expressed in terms of the middle-surface forces, in the following manner:

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{h} \left(\frac{N_x}{E_x} - \nu_y \frac{N_y}{E_y} \right) \\ \epsilon_y &= \frac{1}{h} \left(\frac{N_y}{E_y} - \nu_x \frac{N_x}{E_x} \right) \\ \gamma_{xy} &= \frac{1}{h} \frac{N_{xy}}{G_{xy}} \end{aligned}\right\} \qquad (4.2)$$

Three equations in (4.1a,b) involve four unknown functions in x,y. Hence a fourth equation is needed to determine all the functions. It is obtained by making use of the compatibility equation; it is a relation amongst middle-surface strains and displacements and is of the form:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \qquad (4.3)$$

To reduce the number of equations that must be solved simultaneously, a stress function F(x,y) is introduced. Eqs. (4.1a) will be satisfied if the in-plane forces are defined, as under

$$N_x = h \frac{\partial^2 F}{\partial y^2}, \quad N_y = h \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -h \frac{\partial^2 F}{\partial x \partial y} \tag{4.4}$$

Substituting the relations (4.2) and (4.4) into Eq. (4.3) and the relation (4.4) into Eq. (4.1b), we obtain

$$\frac{1}{E_y} \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - \frac{2\nu_x}{E_x} \right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \tag{4.5}$$

and

$$\frac{D_x}{h} \frac{\partial^4 w}{\partial x^4} + \frac{2(\nu D + D_{xy})}{h} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{D_y}{h} \cdot \frac{\partial^4 w}{\partial y^4} = \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \tag{4.6}$$

We have thus reduced the number of equations that must be solved to two. Eqs. (4.5) and (4.6) are in the modified form, the von Karman large deflection plate equations.

4.3 Approximate Solutions:

Let us consider the simply supported rectangular plate subjected to a uniaxial compressive force N_x. Since a finite-deflection analysis involves the deformations in the plane of middle surface as well as transverse bending deflections, both in-plane and transverse boundary conditions

must be spacified. The transverse boundary conditions corresponding to simply supported edges, i.e.

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \Bigg|_{x=0,a} , \quad w = \frac{\partial^2 w}{\partial y^2} = 0 \Bigg|_{y=0,b} \qquad (4.7)$$

With regard to the in-plane boundary conditions, the following assumptions are made:

1. All edges remain straight and the plate retains its rectangular outline during bending.
2. The shear force N_{xy} vanish along the four edges of the plate.
3. The edges y = 0, a are free to move in y direction.

These boundary conditions are similar to ones stated in Chapter 3.

In the present case the edges are permitted to move, provided that they remain straight. Thus only the average of N_y, and not N_y itself, must be equal to zero, i.e.

$$\int_o^a (N_y)_{y=o,b}\, dx = 0$$

similarly (4.8)

$$\int_o^b (N_x)_{x=o,a}\, dy = - Nb = - \sigma_{xa}\, bh$$

where σ_{xa} is the average stress intensity along x = o,a and the negative sign indicates that the applied load is

compressive.

The first step in the analysis is the choosing of a suitable function for the lateral deflection; we assume that

$$w = f \operatorname{Sin} \frac{m\pi x}{a} \operatorname{Sin} \frac{n\pi y}{b} \tag{4.9}$$

Having assumed the form of w, we next solve Eq. (4.6) for the stress function F. Substitution of (4.9) into (4.6) gives

$$\frac{1}{E_y} \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - \frac{2\nu_x}{E_x} \right) \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 F}{\partial y^4}$$

$$= \frac{f^2}{2} \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 \left[\operatorname{Cos} \frac{2m\pi x}{a} + \operatorname{Cos} \frac{2n\pi y}{b} \right] \tag{4.10}$$

The solution of this equation consists of complementary and a particular part. That is,

$$F = F_c + F_p$$

To obtain the complementary solution, one sets the r.h.s. of the equation equal to zero. But this is equivalent to letting w = 0. Thus the complementary solution of (4.10) corresponds to the in-plane stress distribution that exists in the plate just prior to buckling. At that instant the in-plane stresses are known to consist of a uniform stress $N_x (=N)$ and $N_y = N_{xy} = 0$. Hence

$$F_c = Ay^2 \tag{4.11}$$

In view of the first relation in (4.4) and the fact that $N_x = -\sigma_{xa}h$, Eq. (4.11) can be rewritten as

$$F_c = -\frac{\sigma_{xa}y^2}{2} \tag{4.11a}$$

The particular solution can be of the form

$$F_p = B \operatorname{Cos} \frac{2m\pi x}{a} + C \operatorname{Cos} \frac{2n\pi y}{b} \tag{4.12}$$

Substituting this expression into (4.10) and equating coefficients of the like terms, we obtain

$$B = \frac{E_y f^2}{32}\left(\frac{an}{bm}\right)^2 \quad \text{and} \quad C = \frac{E_x f^2}{32}\left(\frac{bm}{an}\right)^2$$

Thus

$$F_p = \frac{f^2}{32}\left[E_y\left(\frac{an}{mb}\right)^2 \operatorname{Cos} \frac{2m\pi x}{a} + E_x\left(\frac{bm}{an}\right)^2 \operatorname{Cos} \frac{2n\pi y}{b}\right]$$

such that the complete solution of Eq. (4.10) is

$$F = \frac{f^2}{32}\left[E_y\left(\frac{an}{mb}\right)^2 \operatorname{Cos} \frac{2m\pi x}{a} + E_x\left(\frac{bm}{an}\right)^2 \operatorname{Cos} \frac{2n\pi y}{b}\right] - \frac{\sigma_{xa}y^2}{2} \tag{4.13}$$

In order to find out the value of f in Eq. (4.13), which should also satisfy Eq. (4.5), it is necessary to solve Eq. (4.6). Solution of (4.6) by the Galerkin's method gives us (cf.,e.g., Chajes (1974)):

4.4 Effective Width Relation:

As the buckling load is increased beyond the critical load, the distribution of the stresses along the edges becomes progressively nonlinear (see Chapter 1). The loaded edges are more heavily stressed in the vicinity of the unloaded edges and the stresses remain virtually unchanged at the centre of the loaded edge, where the stresses are equal to (or less than) the critical stress σ_{cr}. The effective width b_e is, therefore, defined as

$$b_e \cdot \sigma_{max} = \int_0^b \sigma_x \, dy$$

Substitution for σ_x from (4.10), the above relation yields:

$$\frac{b_e}{b} = \frac{E_x \lambda^4 + E_y}{3E_x \lambda^4 + E_y} \left(1 - \frac{\sigma_{cr}}{\sigma_{max}}\right) + \frac{\sigma_{cr}}{\sigma_{max}} \tag{4.19}$$

or, in terms of the critical load P_o and the applied load P,

$$\frac{b_e}{b} = \frac{E_x \lambda^4 + E_y}{(3E_x \lambda^4 + E_y)(1 - P_o/P) + P_o/P(E_x \lambda^4 + E_y)} \tag{4.20}$$

The form of Eq. (4.20) is similar to the one derived in Chapter 3 Eq. (3.29) . When the ratio b_e/b is computed for a given λ (mb/a) and the ratio E_x/E_y, the values exactly match with those obtained using 'first approximation'

in Chapter 3; these values are shown by dotted curves in Fig. 3-10(a). Hence, to a certain degree of approximation, the method employed in this chapter is similar to the method of successive approach presented in Chapter 3.

CHAPTER V

CONCLUDING REMARKS

This thesis has been concerned with the analysis of buckling and postbuckling behaviour of orthotropic rectangular plates under longitudinal compression. First, a multiple-stiffened light gauge compression member has been idealized as an orthotropic plate, simply supported along the edges. By carrying out an analysis for the buckling strength, a new formula for calculating the 'equivalent plate thickness' has been suggested for design. It turns out that the IS:801-1975 formula grossly over-estimates the equivalent plate thickness.

Whether it is a multiple-stiffened light gauge section or a fibre reinforced composite plate, each can be treated as an orthotropic plate for the purpose of analysis. Postbuckling analysis of such plates has been carried out, first by using the method of successive approximation wherein the displacements are expanded into a power series in terms of an arbitrary parameter, and secondly by an approximate method wherein the differential equation is solved using the Galerkin's method. For these plates, an effective width equation has been derived and curves for load-shortening have been obtained. These curves

show a possibility of a change in the buckling pattern of the plate.

A linear set of equations has been derived to replace the nonlinear large-deflection equations for plate and is shown to have the advantage of simplicity of solution, since much more is known about solving linear partial differential equations than about solving nonlinear partial differential equations. However, the linear set of equations are subject to certain limitations depending upon the application desired. It is to be expected that solution to certain problems might not converge satisfactorily and it appears that the linear equations cannot be used to solve postbuckling problems for plates with initial eccentricities.

The results obtained through second approximation of the present theory agree well with the exact ones for the square plate, at least when $E^* = 1$ (isotropic case). It is to be expected, therefore, that the results for orthotropic plates, presented in this thesis, will be fairly accurate. All the results for plates with finite length-width ratio indicate that the effects of change in the buckling pattern must be considered.

The arbitrary parameter α was taken equal to square root of $(P-P_o)/P_o$. This could just as well have

been related to the shortening or to the centre deflection. For other problems it may be convenient to relate the arbitrary parameter to some other property. For example, in a thermal buckling problem (Stein, 1959), it is related to the average rise in temperature beyond that required for buckling.

For experimental verification, to obtain simply supported loaded edges, as considered in the present investigation, may be impracticable in laboratory. On the other hand, it will be easier to subject the panels to 'flat-end' loading which results in almost complete clamping of the loaded edges. However, if the panel tested is long compared with its width, the size and shape of the buckles near the centre are expected to be almost unaffected by this clamping. Stein (1959) has obtained good correlation between theory and experiment for an aluminium alloy flat plate having length-width ratio of 4. No experimental results for stiffened plates are available.

Suggestions for Further Work:

The following problems can be suggested for further work:

1. Postbuckling analysis of plates having initial geometrical imperfections and/or initial eccentricity in loads.
2. Postbuckling strength of plates in shear.

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